Matching with Overlapping Quotas under Weak Preferences

David Seymour[*], Sinan Ertemel,[†]

November 2019

Abstract

We develop a framework for analyzing a school choice environment where each student may simultaneously satisfy multiple quota criteria, such as race and gender. In our setting, students have weak preferences over their admittance status at a school, allowing students to either prefer one admittance status to another or be indifferent between two admittance statuses. Schools have responsive preferences over students, subject to the quota restrictions. In this environment, a cumulative offer process does not produce stable allocations, as schools may wish to reallocate a student to a different admittance status when new students are considered. Instead, we present a method for analyzing the overlapping quota problem that considers the sets of contracts that students are indifferent between as the main object in the school's choice function, and allows the school to allocate each student to the most desirable status. Our procedure can be nested into a cumulative offer process where students propose sets of contracts they are indifferent between, and schools accept or reject each student's set of contracts. The resulting mechanism is student-optimal, strategy-proof, and respects improvements.

Keywords: matching, school choice, affirmative action, quotas, cumulative offer

JEL: C70, C78, D47, D61, D63

1 Introduction

In school choice, the standard approach is to give equal consideration to all students, regardless of their background. However, schools and policymakers may want to accomplish particular social objectives, such as increasing the number of admitted students in underrepresented groups. Policymakers can use group-specific quotas to increase the number of admitted students from underrepresented groups, using multiple sets of quotas to improve the representation of all the groups simultaneously. If each student has at most

[*]Istanbul Technical University; seymour@itu.edu.tr
[†]Istanbul Technical University, ertemels@itu.edu.tr

one privileged status, the theory of how to incorporate quotas into matching is well-established. Quota seats and non-quota seats are separated, and students with a privileged status have priority over the quota seats (Abdulkadiroğlu, 2005). Any seats that cannot be filled by privileged students are made available to general students (Hafalir et al., 2013). The student-optimal stable allocation can be determined using a cumulative offer process[1] where privileged students have higher priority for seats assigned to their privileged status.

When students can simultaneously hold multiple privileged statuses, the problem is more complicated. Students could be assigned to different statuses at the same school and may either be indifferent between these statuses or may prefer being admitted under a specific status. A desirable mechanism should allocate students to statuses considering the preferences of students that have strict preferences over their admittance status and should allocate students to the most desirable status for the school when students are indifferent between admittance statuses. The overlapping quota problem has been solved for two polar cases, when students are indifferent to their admittance status (Hatfield and Milgrom, 2005; Westkamp, 2013) and when students have a strict preference over admittance status (Aygun and Turhan, 2018). However, none of these methods have natural generalizations that are applicable when students have weak preferences over admittance statuses.

In this manuscript, we provide a framework for analyzing weak preferences that allows students to apply to each school using sets of contracts they are indifferent between. Given a school's set of available contracts, the school's choice correspondence is determined using an iterative process where students contracts are considered on a one-by-one basis. During the process, the school retains all the contracts of conditionally acceptable students, allowing the students' admittance status to adjust to accommodate the statuses of the newly added students. Using this approach, we find the student-optimal stable allocation using a cumulative offer process where students propose sets of contracts that they are indifferent between, and schools conditionally accept sets of contracts rather than individual contracts.

It is natural to assume students have weak preferences over their admittance status. Students are likely to be indifferent between their admittance statuses at the same school when there are no additional advantages of being admitted under a particular status. In college admissions, for instance, many American schools have affirmative action programs that provide no additional funding based on a student's admittance status. A student that is acceptable under multiple statuses is unlikely to care about her admittance status. However, students may have strict preferences over their admittance status when different statuses provide different funding opportunities. The Indian Institutes of Technology (IIT) university admission procedure uses quotas to mitigate historical discrimination faced by disadvantaged castes (Aygün and Turhan, 2017; Sönmez and Yenmez, 2019). As students' opportunities for financial assistance vary based on their admittance status,

[1] The cumulative offer process generalizes the deferred acceptance algorithm (Hatfield and Milgrom, 2005).

they may have strict preferences over their admittance statuses.[2]

There are situations where some students have strict preferences over their admittance status, and others are indifferent, making weak preferences consequential. For instance, in the IIT system, low-income students may be eligible for scholarships when they are admitted under a privileged status, even though wealthier students admitted under the same status may be ineligible. Therefore, some students may prefer privileged seats to general seats, whereas others may be indifferent to their admittance status. Additionally, admission under two different statuses may carry similar opportunities. Some students may be able to distinguish between their opportunities associated with the statuses, whereas others may not, causing some students to have strict preferences over their admittance status and others to be indifferent.

When students have weak preferences over their admittance status, failure to account for these weak preferences leads to suboptimal allocations. A mechanism that incorrectly assumes indifference over admittance statuses leads to inefficient outcomes when some students have strict preference over their admittance status. Because the mechanism does not consider the students' preferences over statuses, it can assign two students to each other's preferred status, leading to a Pareto inferior allocation. Alternatively, if preferences are wrongly assumed to be strict when they are weak, mechanisms that produce stable outcomes under strict preferences fail to satisfy stability under weak preferences. When students that are indifferent between multiple statuses are assigned to the wrong status, a higher-ranked student could be rejected in favor of a lower-ranked student without filling additional privileged seats. This problem is pervasive, a single student that is indifferent between three statuses is incorrectly assigned to a sub-optimal status with two-thirds probability under random tie-breaking. Moreover, if there is a systematic tendency for students with multiple statuses to rank statuses they are indifferent between in a particular order, the students with the higher-ranked status are systematically disadvantaged by the mechanism.[3]

The current matching frameworks for analyzing overlapping quotas do not allow students to have weak preferences over their admittance status. In Hatfield and Milgrom (2005), students are assumed to be indifferent to their admittance statuses. Given a set of available students, a school's choice function is found by maximizing the sum of the students' values where each student's value depends on her rank and whether she holds the status of her assigned seat. The choice function satisfies substitutability and irrelevance of rejected contracts; so, stable allocations are found using a cumulative offer process. When students have weak preferences over statuses, contracts must contain an admittance status; therefore, the method cannot be applied when students have weak preferences over their admittance status. Because the mechanism ignores

[2]Under the IIT mechanism, students are first considered for general seats, so they may have an incentive to reduce their test score (Aygün and Turhan, 2017). Students have targeted their score over a specific range in an attempt to receive a preferable admittance status.

[3]This could occur due to a small tendency for students to inadvertently rank one status over another or because some students intentionally break ties in a way that benefits students of a particular status.

students' preferences over their admittance status, it may sub-optimally assign students to statuses.

Westkamp (2013) provides an alternative mechanism for students that are indifferent to their admittance status. Each status has a capacity and is available to students with the associated status. Statuses are filled sequentially, and any unused capacity is transferred to other statuses. Under the capacity transfer mechanism, students are considered for statuses sequentially without considering their value under other statuses. As a result, the mechanism may produce unstable allocations, giving favorable treatment to students of one status at the expense of other students.

Alternatively, when students have strict preferences over their admittance status, Aygun and Turhan (2018) consider contracts that include a student, school, and admittance status.[4] They use a monotone capacity transfer mechanism, which provides a process for transferring unused capacity to other statuses. Although the school's choice function violates substitutability, it satisfies weaker conditions that allow a cumulative offer process to find the student-optimal stable allocation. Under their structure, the student-optimal cumulative offer process is stable and strategy-proof, and it respects improvement in priorities. However, when students have weak preferences over their admittance status instead of strict preferences, the mechanism violates stability. It rejects higher-ranked student in favor of lower-ranked students without filling any additional privileged seats.

We develop an alternative framework that accommodates students with weak preferences towards their admittance status. In our environment, students have weak preferences over allocations with the same school but have strict preferences over allocations with different schools. Schools have a priority ranking over individual students. Their preferences over groups of students is based on the responsive preferences induced by the school's ranking of individual students (Roth, 1985). Schools choose the maximal set of students subject to the requirement that quota seats are reserved for students with the associated status. Responsive preferences are appealing because the school's choice correspondence is determined using the rankings of individual students without imposing rankings over sets of students. Given their appealing properties, responsive preferences are used in much of school choice, including Abdulkadiroğlu (2005), Hatfield and Milgrom (2005), and Aygun and Turhan (2018).

Responsive preferences are also imposed by existing matching with quota mechanisms. India and Brazil use class and race-based quotas as part of their university admissions process. The German university admissions process has quotas that give priority to top-ranked high school students and students returning after a period of absence (Westkamp, 2013). Under each of these mechanisms, schools have a ranking of students based on the students' test scores that is independent of the students' statuses. A school that

[4]Aygun and Turhan (2018) are motivated by the admission process for engineering schools in India. The system grants priority to various minority groups that overlap. Further, different scholarships may be available to students based on their admittance statuses.

ranks a student higher than another when the students are considered for quota seats also ranks the same student higher for non-quota seats. However, none of these mechanisms currently allow students to have weak preferences over their admittance status.

Our preferences serve as an alternative to slot-specific priorities (Kominers and Sönmez, 2016). In Boston, students that are in walk zones and have siblings that go to the school are given priority for a portion of each school's available seats (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2005). Our preferences are an alternative to slot-specific priorities that do not require a policymaker to specify a priority order over statuses, thereby making it easier to add additional privileged statuses.

We extend the matching with contracts framework (Hatfield and Milgrom, 2005) to account for indifference by using choice correspondences instead of choice functions. The choice correspondences give the set of maximal allocations for students and schools. A student's choice correspondence can include multiple elements because a student may be indifferent between being accepted at a school under two different statuses. Similarly, a school's choice correspondence may have many allocations, as the presence of overlapping quotas allows many different allocations that have the same students. For instance, two students that are allocated to the same school and hold multiple statuses can be allocated to each other's status without changing which students are allocated to the school.

In our environment, contracts that are not in the school's choice correspondence may be in the school's choice correspondence when a new student is added, preventing the cumulative offer process from producing stable allocations. Instead, we reframe the problem to focus on feasible sets of contracts: sets of contracts where all the students in the set of contracts can be allocated to seats at the school. The school's choice correspondence is found by considering students on a one-by-one basis and iteratively determining whether to add the students to the feasible set. When the school prefers a candidate student to the lowest-ranked student in the feasible set that she can replace, all of the candidate student's available contracts are added to the feasible set. The lowest-ranked student's contracts are removed as necessary to ensure that feasibility is maintained. After considering all the students, the students in the final feasible set are in the school's choice correspondence. Further, any allocation of all the students in the feasible set to seats at the school is in the school's choice correspondence.

Given this framework, we propose the Adaptive Assignment Algorithm. Students provide a ranking over sets of contracts, where each set of contracts includes a set of contracts that the student is indifferent between. Students start by applying to their top-ranked school with the first set of contracts. Given the set of contracts that are available to the school, the school chooses the optimal feasible set of contracts and conditionally accepts the students associated with the feasible set. The school retains all the contracts of those students, allowing the school to reallocate the students to different statuses as new students are added to the feasible set. Any student that does not have a contract in the school's choice correspondence is rejected and applies to the next school with the next set of contracts. This process proceeds until all the students are either matched with a school or have no remaining contracts.

Our algorithm has similar properties to the deferred acceptance algorithm introduced in Gale and Shapley (1962). Students are considered on a one-by-one basis and conditionally accepted if they are preferred to the lowest-ranked student they can replace. The algorithm produces the student-optimal stable allocation, is strategy-proof, and respects improvements. Further, our process is linear in the number of students. Alternatively, the method in Hatfield and Milgrom (2005) for determining the student-optimal stable allocation grows exponentially as the number of seats at a school increases. As students being indifferent between their admittance statuses is a special case of students having weak preferences over their admittance status, our algorithm can be used in the Hatfield and Milgrom (2005) environment.

We proceed as follows: we develop the model in Section 2. In Section 3, we develop an algorithm for determining the school's choice correspondence. In Section 4, we look at the properties of the choice correspondence and show that a stable allocation exists. In Section 5, we present the adaptive assignment algorithm. Finally, Section 6 concludes by relating our results to the relevant literature.[5]

2 Model

We use the matching with contracts framework from Hatfield and Milgrom (2005), adopting notation similar to Aygun and Turhan (2018). We assume there are a finite number of students $I = \{i_1, \dots, i_n\}$ and schools $S = \{s_1, \dots, s_m\}$. There is a finite set of statuses $T = \{t_0, t_1, \dots, t_k\}$ that can be held by students, where t_0 is a general status held by all students and $T^p = T \setminus \{t_0\}$ is the set of privileged statuses. The statuses held by the students is given by the correspondence $\tau : I \rightrightarrows T$, where $\tau(i)$ is the set of statuses held by student i.

There are a finite set of possible contracts $X \subseteq I \times S \times T$. Each contract $x \in X$ includes a student $\mathbf{i}(x)$, school $\mathbf{s}(x)$, and assigned status $\mathbf{t}(x)$. Students must hold the status assigned by the contract; therefore, the possible contracts are $X = \{x \in I \times S \times T : \mathbf{t}(x) \in \tau(\mathbf{i}(x))\}$. Given a set of contracts $Y \subseteq X$, the students, schools, and assigned status of Y are given by $\mathbf{i}(Y) \equiv \cup_{x \in Y} \{\mathbf{i}(x)\}$, $\mathbf{s}(Y) \equiv \cup_{x \in Y} \{\mathbf{s}(x)\}$, and $\mathbf{t}(Y) \equiv \cup_{x \in Y} \{\mathbf{t}(x)\}$, respectively. For $Y \subseteq X$, the set of contracts including student i, school s, and student t are $Y_i \equiv \{x \in Y : \mathbf{i}(x) = i\}$, $Y_s \equiv \{x \in Y : s(x) = s\}$, and $Y_t \equiv \{x \in Y : \mathbf{t}(x) = t\}$, respectively. We define sets of contracts that include both student i and school s by $Y_{i,s} = Y_i \cap Y_s$. Similarly, $Y_{s,t} = Y_s \cap Y_t$. Finally, the contracts containing a set of student $I' \subseteq I$ is $Y_{I'} = \{x \in Y : \mathbf{i}(x) \in I'\}$.

[5] A survey of the early matching literature is Roth and Sotomayor (1992). Sönmez and Ünver (2011) survey more recent developments.

School s has a capacity $\bar{q}^s$, where $\bar{q}^s_t$ quota seats are reserved for each privileged status $t \in T^p$. Quota seats are reserved for students of their respective statuses, so $\sum_{t \in T^p} \bar{q}^s_t \leq \bar{q}^s$. The remaining $\bar{q}_{t_0} = \bar{q}^s - \sum_{t \in T^p} \bar{q}^s_t$ seats are available for general students. An allocation is an $X' \subseteq X$ such that $|X'_i| \leq 1$ for all $i \in I$ and $|X'_s \cap X'_t| \leq \bar{q}^s_t$ for all $s \in S$ and $t \in T$. The set of allocations is denoted by $\mathcal{X}$. The set of allocations containing school s is denoted by $\mathcal{X}_s$. We determine the school's choice correspondence for hard bounds. Under hard bounds, any empty privileged seats remain vacant. In the appendix, we consider soft bounds, where any empty privileged seats are available to general students.[6]

2.1 Students' Preferences

Deviating from Aygun and Turhan (2018), we assume student i has a preference relation $\succsim_i$ over X_i. The student has weak preferences over contracts with the same school, but has strict preferences over contracts with different schools. Formally, for $x, x' \in X_i$, if $\mathbf{s}(x) \neq \mathbf{s}(x')$ then $x \nsim_i x'$.[7] A student that prefers to be unmatched to being allocated contract x has $\emptyset \succ_i x$.

The students' preferences allow them to be indifferent between allocations that include the same school but have different statuses. Therefore, we use choice correspondences to account for the students' weak preferences. The choice correspondence for student i is the set of contracts that include student i and are maximal for student i under $\succsim_i$:

$$\mathcal{C}_i(Y) = \{x \in Y_i : \mathbf{s}(x) \succsim_i s' \text{ for all } s' \in \mathbf{s}(Y_i) \cup \emptyset\}$$

As students strict have preferences over contracts with different schools, $\mathbf{s}(x) = \mathbf{s}(x')$ for all $x, x' \in \mathcal{C}_i(Y)$.

2.2 Schools' Preferences

Each school has a strict priority order over pairs of students, denoted by π_s, that is independent of the students' statuses. Student i has a higher priority than student i' at school s if $i \, \pi_s \, i'$. Student i is acceptable to school s if $i \, \pi_s \, \emptyset$ and is unacceptable if $\emptyset \, \pi_s \, i$. The priority order profile of the schools is $\Pi = (\pi_{s_1}, \ldots, \pi_{s_m}\}$.

Given the priority order over students, schools preferences are responsive with respect to π_s. Preferences are responsive when adding acceptable students, removing unacceptable students, and replacing lower-ranked students with higher-ranked students all lead to preferred allocations. Formally, the binary relation $\succsim_s$ is

[6] The theory behind hards bounds is more elegant and has a simpler algorithm for determining the school's choice correspondence; however, it is also necessary to consider soft bounds due to the inefficiency resulting from the empty seats that may occur under hard bounds. For soft bounds, we use minority reserves due to the adverse effect that majority quotas can have on the privileged groups (Kojima, 2012; Hafalir et al., 2013).

[7] This requirement is only over individual contracts. It does not require students to prefer all the contracts of one school above another. For instance, if t_1 is a status with funding and t_0 is a status without funding, student i may have $(s, t_1) \succ_i (s', t_1) \succ_i (s, t_0)$.

responsive if it is reflexive, transitive, and satisfies the following conditions:

i. If $\mathbf{i}(X'') = \mathbf{i}(X') \cup \{i\}$ for $i \notin \mathbf{i}(X')$ and $i\pi_s\emptyset$, then $X'' \succsim_s X'$

ii. If $\mathbf{i}(X'') = \mathbf{i}(X') \setminus \{i\}$ for $i \in \mathbf{i}(X')$ and $\emptyset\pi_s i$, then $X'' \succsim_s X'$

iii. If $\mathbf{i}(X'') = \mathbf{i}(X') \cup \{i\} \setminus \{i'\}$ for $i' \in \mathbf{i}(X')$ and $i \notin \mathbf{i}(X')$ and $i'\pi_s i$, then $X'' \succsim_s X'$

The school's ranking over students induces a preorder $\succsim_s$ on $\mathcal{X}_s$.

Given a set of contracts Y, an allocation $X' \subseteq Y_s$ is in the schools choice correspondence if it is maximal under the preorder $\succsim_s$. Although, allocations need not be comparable,[8] we show in Proposition 4 that any allocation in the choice correspondence of Y is weakly preferred to any other allocations contained in Y. Therefore, the choice correspondence for school s:

$$\mathcal{C}_s(Y) = \{X' \in Y_s : X' \succsim_s X'' \;\forall\; X'' \in Y_s\}$$

is well-defined and consists of all maximal elements of Y_s under the preorder $\succsim_s$. As the school is indifferent between all the contracts in the choice correspondence, $\mathbf{i}(X') = \mathbf{i}(X'')$ for all $X', X'' \in \mathcal{C}_s(Y)$.[9]

2.3 Equilibrium

We focus on stable allocations, allocations that are individually rational (IR) and unblocked (UB). An allocation is individually rational when the all the students' and schools' contracts are acceptable. An allocation is unblocked when there is no allocation that is strictly preferred to the original allocation by a school and weakly preferred to the original allocation by the new students allocated to the school.

Definition 1. *An allocation X' is* ***stable*** *if it is IR and UB:*

IR: $X'_i \in \mathcal{C}_i(X')$ *for all* $i \in I$ *and* $X'_s \in \mathcal{C}_s(X')$ *for all* $s \in S$.

UB: There is no $s \in S$ *and* $X'' \subseteq X_s$ *such that* $X'' \in \mathcal{C}_s(X' \cup X'')$, $X'_s \notin \mathcal{C}_s(X' \cup X'')$, *and* $X''_i \in \mathcal{C}_i(X' \cup X'')$ *for* $i \in \boldsymbol{i}(X'')$.

Stability accounts for student's weak preferences by allowing for blocking coalitions where students are indifferent between the original allocation and a new allocation. Example 1 illustrates that a student in a blocking coalition can be indifferent between the original allocation and a new allocation.

[8]As there are many different preferences that are consistent with a given priority order, we assume that two allocation are not comparable if the preferences over the two allocations cannot be determined by π_s and responsiveness. This modeling assumption makes the proofs easier without affecting the results.

[9]This result is also shown in Propositions 4.

Example 1. *Let* $I = \{i_1, i_2, i_3\}$*,* $S = \{s\}$*, and* $T = \{t_0, t_1\}$*. Let* $\bar{q}^s = 2$ *and* $\bar{q}^s_{t_1} = 1$*. Assume* $\tau(i_1) = \tau(i_3) = \{t_0, t_1\}$ *and* $\tau(i_2) = \{t_0\}$*. The school's preferences are* $i_1 \,\pi_s\, i_2 \,\pi_s\, i_3$*. The students' preferences are* $(i_1, t_0) \sim_{i_1} (i_1, t_1) \succ_{i_1} \emptyset$*,* $(i_2, t_0) \succ_{i_2} \emptyset$*, and* $(i_3, t_1) \succ_{i_3} (i_3, t_0) \succ_{i_3} \emptyset$*. Then* $X' = \{(i_1, t_0), (i_3, t_1)\}$ *is blocked by* $X'' = \{(i_1, t_1), (i_2, t_0)\}$*.*

3 Hard Bounds

Given a set of contracts Y, the choice correspondence of school s selects the set of maximal allocations that are contained in Y_s. Maximal allocations cannot be found by considering contracts iteratively, as in the standard cumulative offer process. When a new student is considered, a school may want to reallocate some of the conditionally accepted students to different statuses. To choose the highest-ranked students conditional on satisfying the quota, a school may want to reallocate a student to a previously rejected contract. Example 2 shows that contracts that are not in the choice correspondence at one stage may become acceptable to the school when a new contract is added to the set of contracts that is considered.

Example 2. *Let* $I = \{i_1, i_2, i_3\}$*,* $S = \{s\}$*, and* $T = \{t_0, t_1\}$*. Let* $i_1 \,\pi\, i_2 \,\pi\, i_3$*. Assume* $\bar{q}^s_{t_0} = \bar{q}^s_{t_1} = 1$*. Let* $Y_1 = \{(i_1, t_0), (i_1, t_1), (i_3, t_1)\}$ *and* $Y_2 = Y_1 \cup \{(i_2, t_0)\}$*. Then* $\mathcal{C}(Y_1) = \{(i_1, t_0), (i_3, t_1)\}$*; however,* $\mathcal{C}(Y_2) = \{(i_1, t_1), (i_2, t_0)\}$ *and* $\mathcal{C}(\mathcal{C}(Y_1) \cup \{(i_2, t_0)\}) = \{(i_1, t_0), (i_3, t_1)\}$*.*

In this example, the second-ranked student cannot directly replace the third-ranked student when only acceptable contracts are maintained. However, changing the status of the highest-ranked student to a previously rejected contract allows the second-ranked student to replace the third-ranked student. The only types of rejected contracts that schools wish to maintain are the contracts of students that have other contracts in the school's choice correspondence. Therefore, we reframe the problem to allow schools to hold all the contracts of their conditionally accepted students. Student without any contracts in the choice correspondence are not worth reconsidering in subsequent stages, allowing all the students' contracts to be rejected.

Formally, we consider a set of students $I' \subseteq \mathbf{i}(Y_s)$ and their corresponding set of available contracts $Z' = \bigcup_{i \in I'} Y_{i,s}$. We determine whether the students in I' can be allocated to their schools using the contracts in Z'. To determine whether the students with contracts in Z' are in the choice correspondence of school s, we consider students on a one-by-one basis. Given a set of admissible students Z', the school benefits by adding acceptable student i with contracts $Y_{i,s}$ when it can add the student without replacing another student or replace a lower-ranked student in Z'. When the student replaces another student, it is optimal for the school to replace the lowest-ranked student possible. Given this setup, we can characterize the school's choice correspondence. A set of students is in the school's choice correspondence if (i) the seats can be allocated to the students, (ii) the students are acceptable to the school, and (iii) there are no feasible replacements that lead to a preferred set of students.

3.1 Feasibility

Given a set of contracts $Z' \subseteq X$, we determine whether there is an allocation $X' \subseteq Z'$ of all the students in $\mathbf{i}(Z')$ to school s. To this end, we define the feasibility of Z' in a recursive manner. A set of contracts is feasible for a set of statuses T' for school s when (i) there are enough seats at school s of statuses contained in T to accommodate the students whose contracts are contained T' and (ii) there are enough seats at school s of statuses contained in T to accommodate the students whose contracts are contained T'' for any $T'' \subset T'$. A set of contracts is full for T' if it is feasible and it is impossible to add another student whose contracts are contained in T' without violating feasibility. Formally,

Definition 2. *Let $Z' \subseteq X_s$ and let $T' \subseteq T$. A set of contracts Z' is*

(i) ***feasible*** *for T' if $|\{i \in \boldsymbol{i}(Z') : \boldsymbol{t}(Z'_i) \subseteq T''\}| \leq \sum_{t \in T''} \bar{q}^s_t$ for all $T'' \subseteq T'$.*

(ii) ***full*** *for T' if Z' is feasible for T' and $|\{i \in \boldsymbol{i}(Z') : \boldsymbol{t}(Z'_i) \subseteq T'\}| = \sum_{t \in T'} \bar{q}^s_t$.*

We say that a set of contracts Z' is feasible if Z' is feasible for T.

When a set of contracts Z' is feasible for T', the students of $\mathbf{i}(Z')$ whose contracts are contained in T' can be allocated to school s. It is necessary to examine the subsets $T'' \subseteq T'$ because students with contracts contained in T'' can only be allocated to seats with statuses contained in T''. Therefore, when it is impossible to allocate these students to seats in T'', these students cannot be allocated to seats in T'. Further, when determining whether a set is feasible for T', we disregard students with $\mathbf{t}(Z'_i) \nsubseteq T'$ because these students can be allocated to statuses outside T'. A feasible set of contracts Z' is full for T' when adding another student with contracts contained in T' violates feasibility. For a given Z', the set of full statuses determines whether a student can be added to Z' without replacing another student. When a student needs to replace another student, the set of full statuses determines which student in Z' the student can replace while maintaining feasibility. The following example illustrates feasible and full sets of contracts.

Example 3. *Assume that $I = \{i_1, i_2, i_3\}$, $S = \{s\}$, $T = \{t_0, t_1\}$, $\bar{q}_{t_0} = 2$, and $\bar{q}_{t_1} = 1$. Further, let $Z' = \{(i_1, t_0), (i_2, t_1), (i_3, t_0), (i_3, t_1)\}$, Z' is full for $\{t_1\}$ and T, and Z' is feasible but not full for $\{t_0\}$. $X' = \{(i_1, t_0), (i_2, t_1), (i_3, t_0)\}$ assigns student in $\boldsymbol{i}(Z')$ to seats in T. When $Z'' = \{(i_1, t_0), (i_2, t_1), (i_3, t_1)\}$, Z'' is not feasible for both $\{t_1\}$ and T. Here, $|\{i \in \boldsymbol{i}(Z'') : \boldsymbol{t}(Z''_i) \subseteq \{t_1\}\}| = 2$ but $\bar{q}_{t_1} = 1$. So, there is no allocation $X'' \subseteq Z''$ with $\boldsymbol{i}(X'') = \boldsymbol{i}(Z'')$.*

In the example, the set Z' is full for T, so it is possible to allocate the students to seats at the school. However, by removing an available contract, the new set of contracts Z'' is not feasible for $\{t_1\}$. Because there is only one seat available under status t_1, it is impossible to assign both i_2 and i_3 to t_1; therefore, it is there is no allocation of all the students in Z' to the school's available seats.

These results generalize beyond the example. When Z' is feasible for T', it is possible to allocate the students with $\mathbf{t}(Z'_i) \subseteq T'$ to the seats in T'. Therefore, when Z' is feasible, it is possible to allocate all the students in $\mathbf{i}(Z')$ to seats at the school. However, when Z' is not feasible for some T', Z' is not feasible and it is impossible to allocate the students in $\mathbf{i}(Z')$ to seats at the school. Formally,

Proposition 1. *$Z' \subseteq X_s$ is feasible for $T' \subseteq T$ if and only if there is an allocation $X' \subseteq Z'$ where $\boldsymbol{i}(X') = \{i \in \boldsymbol{i}(Z') : \boldsymbol{t}(Z'_i) \subseteq T'\}$.*

The proof of this and subsequent results are provided in Appendix A. An outline of the proof follows each proposition. We use induction to show that the feasibility of a set of contracts implies the existence of allocations. When Z' is feasible for $T' = \{t\}$, the number of students with $\mathbf{t}(Z'_i) = \{t\}$ is less than or equal to q_t^s; therefore, the students of status t can be allocated to the school under their contracts. When the results hold for all feasible statuses of size up to size N, then for any T' of size $N+1$ we can construct an allocation through an iterative process. Starting with Z', we iteratively remove contracts. In each iteration we first check whether there is some $T'' \subset T'$ that is full. If not, we choose a student i with Z_i that contains multiple contracts and remove all but one of their contracts from Z'. We repeat this process until either all the students have a single contract or there is a full $T'' \subset T'$. When all the students have a single contract, this set of contracts is an allocation. Alternatively, when there is a full $T'' \subset T'$, we can assign the students with statuses contained in T'' to seats in T'' using the inductive step. The remaining students can be assigned seats in $T' \setminus T''$. Conversely, when there is allocation $X' \subseteq Z'$. For any T', only students with $\mathbf{t}(Z'_i) \subseteq T'$ can be allocated to seats in T'. Therefore, the number of students with contracts contained in any $T' \subseteq T$ is at most $\sum_{t \in T'} \bar{q}_t^s$, so feasibility is satisfied.

3.2 Replaceability

Having characterized feasibility, we examine the process of adding student i to Z'. When student i has a set of available contracts $Y_{i,s}$, student i can be added without replacing any students when $Z' \cup Y_{i,s}$ is feasible. Alternatively, when $Z' \cup Y_{i,s}$ is not feasible, student i can replace student i' when $Z' \cup Y_{i,s} \setminus Z'_{i'}$ is feasible. The minimum full set of contracts of Z' containing the statuses of $Y_{i,s}$ is critical for determining whether it is necessary to replace a student when adding i and which students i can replace. We formalize the minium full set of contracts using the binding set.

Definition 3. *Assume $Z' \subseteq X_s$ is feasible. The statuses T' are **binding statuses** for $Y_{i,s}$ under Z' if*

(i) $\boldsymbol{t}(Y_{i,s}) \subseteq T'$

(ii) Z' *is full for* T'

(iii) There is no $T'' \subset T'$ *such that* $\boldsymbol{t}(Y_{i,s}) \subseteq T''$ *and* Z' *is full for* T''

If there is no T' that is binding, we say that $Y_{i,s}$ is unbounded under Z'.

If the binding set T' exists for $Y_{i,s}$ under Z', the **binding contracts** are $B(Y_{i,s}, Z') = \bigcup\{Z'_i : \mathbf{t}(Z'_i) \subseteq T'\}$. The binding contracts are the contracts in Z' of students whose statuses are contained in T'. Using the binding contracts, the binding statuses are $T' = \mathbf{t}(B(Y_{i,s}, Z'))$ and the **binding students** are $\mathbf{i}(B(Y_{i,s}, Z'))$. If there is no T' that is full and satisfies $\mathbf{t}(Y_{i,s}) \subseteq T'$, then $Y_{i,s}$ is unbound under Z' and $B(Y_{i,s}, Z') = \infty$.[10]

The binding set of $Y_{i,s}$ determines whether student i can be added to the school without needing to replace another student. When $B(Y_{i,s}, Z') = \infty$, the contracts of a student i can be added without replacing any students in $\mathbf{i}(Z')$, so $Z' \cup Y_{i,s}$ is feasible. However, when $B(Y_{i,s}, Z') \neq \infty$, $Z' \cup Y_{i,s}$ is not feasible and student i cannot be added to the allocation without replacing another student. These properties are illustrated in Example 4.

Example 4. *Let $I = \{i_1, i_2\}$, $S = \{s\}$, $T = \{t_0, t_1\}$, and $\bar{q}_{t_0} = \bar{q}_{t_1} = 1$. Let $Z' = \{(i_1, t_0)\}$, $Y_i = \{(i_2, t_0)\}$, and $Y'_i = \{(i_2, t_0), (i_2, t_1)\}$. Then Z' is feasible and is full for $\{t_0\}$. The allocation for Z' is $X' = \{(i_1, t_0)\}$. Here, $B(Y_i, Z') = \{t_0\}$. Both $\{t_0\}$ and T are violated by $Z'' = Z' \cup Y_i$; therefore, there is no allocation of students in $X'' \subseteq Z''$ with $\boldsymbol{i}(X'') = \boldsymbol{i}(Z'')$. On the other hand, $B(Y'_i, Z') = \infty$; therefore, $Z''' = Z' \cup Y'_i$ is feasible with the allocation $X'' = \{(i_1, t_0), (i_2, t_1)\}$.*

When adding $Y_{i,s}$ to Z' is infeasible, the binding set also determines which replacements are feasible. Student i can replace a student $i' \in \mathbf{i}(Z')$ if and only if the contracts of student i' are contained in $B(Y_{i,s}, Z')$. When $i \in \mathbf{i}(B(Y_{i,s}, Z')$, the new set of contracts $Z'' = Z' \cup Y_{i,s} \setminus Z'_{i'}$ is feasible, and the students $\mathbf{i}(Z'')$ can be allocated to the school. The binding statuses can contain statuses outside student i's status, allowing a student to replace another student even when their statuses do not intersect. Example 5 illustrates that the binding set depends on the set of contracts offered through $Y_{i,s}$ and that when the binding set is larger than the set of statuses in $\mathbf{t}(Y_{i,s})$ the contracts of other students can change to accommodate student i.

Example 5. *Assume that $I = \{i_1, i_2, i_3\}$, $S = \{s\}$, and $T = \{t_0, t_1\}$. Let $\bar{q}_{t_0} = 1$ and $\bar{q}_{t_1} = 1$. Assume $Z' = \{(i_1, t_0), (i_2, t_0), (i_2, t_1)\}$, $Y' = \{(i_3, t_0)\}$, and $Y'' = \{(i_3, t_1)\}$. Then Z' is feasible, and $\{t_0\}$ and T are full. The allocation under Z' is $X' = \{(i_1, t_0), (i_2, t_1)\}$. As $B(Y', Z') = \{t_0\}$, i_3 can only replace i_1 using the*

[10] The binding set of statuses and contracts are well-defined. We show this result in Appendix B.

contract in Y'. $Z'' = Z' \cup Y' \setminus Z'_{i_1}$ is feasible with allocation $X'' = \{(i_2, t_1), (i_3, t_0)\}$. As $B(Y'', Z') = T$, i_3 can replace i_1 or i_2 using the contract in Y''. Under $Z''' = Z' \cup Y'' \setminus Z'_{i_1}$ the allocation $X''' = \{(i_2, t_0), (i_3, t_1)\}$ and i_2's admitted status changes from t_1 to t_0.

The results illustrated in Examples 4 and 5 are generalized in Proposition 2, below.

Proposition 2. *Assume $Z' \subseteq X_s$ is feasible and let $i \notin \boldsymbol{i}(Z')$ with $Y_{i,s} \neq \emptyset$*

(i) $Z' \cup Y_{i,s}$ is feasible if and only if $B(Y_{i,s}, Z') = \infty$.

(ii) If $B(Y_{i,s}, Z') \neq \infty$, then $Z' \cup Y_{i,s} \setminus Z'_{i'}$ is feasible if and only if $i' \in \boldsymbol{i}(B(Y_{i,s}, Z'))$.

When $Z' \cup Y_{i,s}$ is feasible for T' with $\mathbf{t}(Y_{i,s}) \subseteq T'$, removing $Y_{i,s}$ decreases the number of students with statuses contained in T' by one; therefore, none of these sets are full. As there are no full sets of Z' containing $\mathbf{t}(Y_{i,s})$, $B(Y_{i,s}, Z') = \infty$. Alternatively, when $B(Y_{i,s}, Z') = \infty$, there are no full sets T' where $\mathbf{t}(Y_{i,s}) \subseteq T'$. Adding $Y_{i,s}$ to Z' increases the number of students whose contracts are contained in T' by one; so T' becomes at most full. For any T' where $\mathbf{t}(Y_{i,s}) \nsubseteq T'$, adding $Y_{i,s}$ to Z' does not change which students have contracts contained in T', so these sets remain feasible. As a result, $Z' \cup Y_{i,s}$ is feasible.

When $B(Y_{i,s}, Z') < \infty$ and $Z' \cup Y_{i,s} \setminus Y_{i,s}$ is feasible, the contracts of student i' must be in the binding set. Otherwise, removing student i' leaves the number of student in the binding set unchanged, whereas adding student i increases the number of student in the binding set by one. As $B(Y_{i,s}, Z')$ is full for Z', replacing student i' by student i causes the number of students in the binding set to exceed the number of available seats, contradicting the feasibility of $Z' \cup Y_{i,s} \setminus Y_{i,s}$. Alternatively, when student i with contract $Y_{i,s}$ replaces student $i' \in \mathbf{i}(B(Y_{i,s}, Z'))$, the number of students whose contracts are contained in T' can only increase for T' that contains $\mathbf{t}(Y_{i,s})$ but do not contain $\mathbf{t}(Z'_{i'})$. However, any full set that contains $\mathbf{t}(Y_{i,s})$ but does not contain $\mathbf{t}(Z'_{i'})$ would contain $B(Y_{i,s}, Z')$, contradicting $i' \in \mathbf{i}(B(Y_{i,s}, Z'))$. Therefore, any sets have more students under Z'' than under Z' are not full under Z'. So, $Z' \cup Y_{i,s} \setminus Y_{i,s}$ is feasible.

3.3 Choice Correspondence

We characterize the optimal replacement strategy for a school s that considers students on a one-by-one basis. The school has a feasible set of contracts Z' and considers adding student i with contracts $Y_{i,s}$. As schools preferences are responsive, the school adds student i when she is acceptable and her contracts can be added to Z' without needing to replace another student. When a student needs replace another student, the school replaces the lowest-ranked student possible provided the new student is preferred to the student that is replaced. We call the lowest-ranked student that student i can replace the minimum admissible student:

Definition 4. *Let $Z' \subseteq X_s$ be feasible, and let $i \in \boldsymbol{i}(Y_s) \setminus \boldsymbol{i}(Z')$. The* ***minimum admissible student*** *is*

$$i^*(Y_{i,s}, Z') = \begin{cases} \min_{\pi_s} \boldsymbol{i}(B(Y_{i,s}, Z')) & \text{if } B(Y_{i,s}, Z') \neq \infty \\ \emptyset & \text{if } B(Y_{i,s}, Z') = \infty \end{cases}$$

The minimum admissible student is the lowest-ranked student in Z' whose contracts are contained in $B(Y_{i,s}, Z')$. If the binding set is unbounded, there are some unoccupied seats and student i can be admitted without replacing another student. When the binding set is bounded, the minimum admissible student is the lowest-ranked student that is replaceable by i. The following result states that it is optimal for the school to replace the minimum admissible student when the candidate student is preferred to the minimum admissible student.

Proposition 3. *Assume $Z' \subseteq X_s$ is feasible and $i'\pi_s\emptyset$ for all $i' \in \boldsymbol{i}(Z')$. For $i \notin \boldsymbol{i}(Z')$ and $Y_{i,s} \neq \emptyset$, define $i^* = i^*(Y_{i,s}, Z')$. If $i'\pi_s i^*$ then there is an allocation $X'' \subseteq Z'' = Z' \cup Y_{i',s} \setminus Z'_{i^*}$ such that (i) $\boldsymbol{i}(X'') = \boldsymbol{i}(Z'')$, and (ii) $X'' \succ_s X'$ for any allocation $X' \subseteq Z' \cup Y_{i',s}$ where $\boldsymbol{i}(X'') \neq \boldsymbol{i}(X')$.*

When $i^* = \emptyset$ and $i\pi_s i^*$, Z'' is feasible with an allocation $X'' \subseteq Z''$ where $\mathbf{i}(X'') = \mathbf{i}(Z'')$. Any allocation $X' \subseteq Z' \cup Y_{i,s}$ with $\mathbf{i}(X') \neq \mathbf{i}(X'')$ has $\mathbf{i}(X') \subset \mathbf{i}(X'')$. Because schools have responsive preferences, X'' is preferred to X'. When $i^* \neq \emptyset$ and $i\pi_s i^*$, Z'' is preferred to Z' as it replaces student i^* by student i. Further, any other allocation must be contained in $Z' \cup Y_{i,s} \setminus Z'_{i'}$ for some $i' \in \mathbf{i}(B(Y_{i',s}, Z'))$. When these allocations have the maximal number of students, they contain $\mathbf{i}(Z') \cup i \setminus i'$ for some $i' \in \mathbf{i}(B(Y_{i',s}, Z'))$. The allocation with $i' = i^*$ removed is preferred the allocations with any other $i' \in B(Y_{i,s}, Z')$ removed. Further, Any allocation that does not have the maximal number of students, is contained in an allocation X''' with students $\mathbf{i}(Z') \cup i \setminus i'$ for some $i' \in B(Y_{i,s}, Z')$, so $X'' \succ_s X''' \succsim_s X'$.

The minimum admissible student provides a concise way to determine whether a set of contracts is in the choice correspondence of school s, given a set of available contracts Y_s. By considering a set of students I' and the corresponding contracts $Z' = \bigcup_{i \in I'} Y_{i,s}$, we can determine whether it is possible to allocate the students in Z' to seats at a school, and whether the corresponding allocation is in the school's choice correspondence. Specifically, a set of students and their associated contracts are the school's choice correspondence if and only if the set of contracts are feasible, there are no students that are unacceptable, and there are no beneficial replacement by a student in $\mathbf{i}(Y_s) \setminus \mathbf{i}(Z')$.

Proposition 4. *Let $Y \subseteq X$ and let $Z' = \{x \in Y_s : \boldsymbol{i}(x) \in I'\}$ for some $I' \subseteq I$. $X' \in \mathcal{C}_s(Y)$ for some $X' \subseteq Z'$ such that $\boldsymbol{i}(X') = \boldsymbol{i}(Z')$ if and only if Z' satisfies the following conditions:*

(i) Feasibility: Z' is feasible.

(ii) Acceptability: $i \, \pi_s \, \emptyset$ *for all* $i \in \boldsymbol{i}(Z')$.

(iii) No beneficial replacement: $i^*(Y_{i,s}, Z')\pi_s i$ *for all* $i \in \boldsymbol{i}(Y_s) \setminus \boldsymbol{i}(Z')$.

If X' is in the schools choice correspondence, then X' is an allocation; therefore, the associated set of contracts $Z' = \bigcup_{i \in I'} Y_{i,s}$ is feasible. An allocation with unacceptable student can be improved upon by removing unacceptable students; so, there are no unacceptable students in $\mathbf{i}(Z') = \mathbf{i}(X')$. For an allocation to be in the choice correspondence, there are no students that are not in the allocation but are preferred to the minimum admissible student; otherwise, the school can choose a preferred allocation by adding the student to its allocation and removing the minimal admissible student.

Alternatively, the poset $\succsim_s$ has a set of maximal elements under $\succsim_s$ which satisfy conditions (i), (ii), and (iii). When the maximal Z' have the same students, their allocations are equivalent and are preferred to every other allocation. On the contrary, if there were $Z'' \neq Z'''$ that satisfied conditions (i), (ii), and (iii), then there is a highest-ranked student among the students that are only in one of the two sets of contracts. This student can replace a lower-ranked student in the other set of contracts, contradicting the no beneficial replacement condition. Thus, Z' is unique and the associated X' is in the school's choice correspondence.

The proof of Proposition 4 implies that any set of contracts in the schools choice correspondence is preferred to any other allocation of students in the choice set. Therefore, it is unnecessary to impose additional assumptions on the preferences of schools beyond the minimal relationship implied by responsiveness, transitivity, and the ordering over students π_s. Further, every allocation in the choice correspondence contains the same set of students. Therefore, given a set of available contracts, the choice correspondence chooses a unique set of students.

Corollary 1. *Let* $Y \subseteq X$. *For any* $X' \in \mathcal{C}_s(Y)$ *and allocation* $X'' \subseteq Y$, $X' \succsim X''$. *Further, for any* $X', X'' \in \mathcal{C}_s(Y)$, $\boldsymbol{i}(X') = \boldsymbol{i}(X'')$.

To determine the choice correspondence of school s, we follow an iterative process. Starting with $Z' = \emptyset$, we go through the school's available contracts on a student-by-student basis. If the candidate student is preferred to the minimum admissible student, then the candidate student's contracts are added to Z' and the minimum admissible student's contracts are removed. Once all the students are considered, the students in $\mathbf{i}(Z')$ are the students in the schools choice correspondence.

Algorithm 1. *Let* $Y \subseteq X$ *with* $\boldsymbol{i}(Y_s) = \{i_j\}_{j \in \{1,\ldots,|\boldsymbol{i}(Y_s)|\}}$. *Let* $Z^0 = \emptyset$. *For each* $j \in \{1, \ldots, |\boldsymbol{i}(Y_s)|\}$, *let*

$i^* = i^*(Y_{i_j,s}, Z^{j-1})$.

$$Z^j = \begin{cases} Z^{j-1} \cup Y_{i_j,s} \setminus Y_{i^*,s} & \textit{if } i_j \, \pi_s \, i^* \\ Z^{j-1} & \textit{otherwise} \end{cases}$$

The final set of contracts is $Z' = Z^{|\boldsymbol{i}(Y_s)|}$.

Proposition 5. *The set of contracts* Z' *chosen by Algorithm 1 is unique and satisfies* $\boldsymbol{i}(\mathcal{C}_s(Y)) = \boldsymbol{i}(Z')$.

In the algorithm, students are considered on a one-by-one basis. When a student is added to Z^j, the minimal admissible student is removed to maintain feasibility. The candidate student's contracts are added to the Z^j when they are preferred to the minimal admissible student, so the students in Z^j are acceptable. At each stage in the algorithm, the replacement is optimal and causes the overall student ranking to improve as the algorithm proceeds. Therefore, any student that is rejected at some stage of the algorithm is worse than the minimum admissible student when the algorithm terminates. So, the conditions of Proposition 4 are satisfied and the students in $\mathbf{i}(Z')$ are in the schools choice correspondence.

4 Properties

We examine the properties of the schools' choice correspondences by expanding the definitions of irrelevance of rejected contracts, substitutability, unilateral substitutability, and bilateral substitutability to account for the schools' weak preferences over allocations. These definitions coincide with their definitions for choice functions when the choice correspondences consist of a single set of contracts.

Definition 5. *A choice correspondence* $\mathcal{C}$ *satisfies* **irrelevance of rejected contracts (IRC)** *when for all* $Y \subset X$ *and* $x \in X \setminus Y$*, if* $x \notin X'$ *for all* $X' \in \mathcal{C}(Y \cup \{x\})$ *then* $\mathcal{C}(Y) = \mathcal{C}(Y \cup \{x\})$.

A choice correspondence satisfies IRC when the set of allocations in the schools choice correspondence is unaffected by the addition of a contract that is not in any allocation of the schools choice correspondence. When students have strict preferences over contracts, this condition is necessary to ensure that a stable allocation exists (Aygün and Sönmez, 2013). Our environment satisfies IRC.

Proposition 6. *The choice correspondence* $\mathcal{C}_s$ *satisfies IRC.*

For $\mathcal{C}_s$ the set of chosen allocations depend on the pairwise comparison of allocations using $\succsim_s$. Allocations in the original choice set continue to be preferred to all the original allocations, and no allocations with the irrelevant contract are in the choice correspondence. Therefore, the sets of contracts in the choice correspondence are the same, regardless of whether the irrelevant contract is included.

Definition 6. *A school's choice correspondence* $\mathcal{C}_s$ *satisfies*

(i) **substitutability** *when for all* $x, x' \in X$ *and* $Y \subset X$*, if* $x \notin X'$ *for any* $X' \in \mathcal{C}_s(Y \cup \{x\})$ *then* $x \notin X'$ *for any* $X' \in \mathcal{C}_s(Y \cup \{x, x'\})$*.*

(ii) ***unilateral substitutability (ULS)*** *when for all* $x, x' \in X$ *and* $Y \subset X$ *such that* $\boldsymbol{i}(x) \notin \boldsymbol{i}(Y)$*, if* $x \notin X'$ *for any* $X' \in \mathcal{C}_s(Y \cup \{x\})$ *then* $x \notin X'$ *for any* $X' \in \mathcal{C}_s(Y \cup \{x, x'\})$*.*

(iii) **bilateral substitutability (BLS)** *when for all* $x, x' \in X$ *and* $Y \subset X$ *such that* $\boldsymbol{i}(x), \boldsymbol{i}(x') \notin \boldsymbol{i}(Y)$*, if* $x \notin X'$ *for any* $X' \in \mathcal{C}_s(Y \cup \{x\})$ *then* $x \notin X'$ *for any* $X' \in \mathcal{C}_s(Y \cup \{x, x'\})$*.*

These substitutability conditions are successively weaker conditions that can be combined with IRC to ensure the existence of a stable allocation when students have strict preferences over their admittance status. Our choice correspondence violates substitutability and ULS, but satisfies BLS.

Proposition 7. *The school's choice correspondence,* $\mathcal{C}_s$*, violates substitutability and ULS, but satisfies BLS.*

Example 6, below, violates ULS; therefore, it violates substitutability. ULS is violated because the addition of a contract changes the schools preferred assignment of one student's status to a status that is not in the original choice correspondence. Alternatively, Algorithm 1 shows that BLS is satisfied. Any contract that is rejected at some stage of the choice correspondence algorithm is not acceptable when a contract of an additional students is considered in the next stage of the algorithm.

Example 6. *Let* $I = \{i_1, i_2, i_3\}$*,* $S = \{s\}$*,* $T = \{t_0, t_1\}$*. Let* $i_1 \,\pi\, i_2 \,\pi\, i_3$*. Assume* $\bar{q}^s = 2$ *and* $\bar{q}^s_{t_0} = \bar{q}^s_{t_1} = 1$*. Let* $Y = \{(i_1, t_0), (i_3, t_1)\}$*,* $x = (i_2, t_0)$*, and* $x' = (i_1, t_1)$*. Then* $\mathcal{C}_s(Y \cup \{x\}) = Y$ *but* $\mathcal{C}_s(Y \cup \{x\}) = \{x, x'\}$*.*

The results from Hatfield and Kojima (2010) and Aygün et al. (2012) show that when the schools' choice functions satisfy IRC and BLS, there exists a stable allocation. However, this result does not directly apply to our framework, due to the possibility that schools and students are indifferent between different contracts.[11] However, the following result shows that there exists a stable allocation.

Proposition 8. *There exists a stable allocation.*

To show that a stable allocation exists, we create quasi-contracts that group the contracts that students are indifferent between into one quasi-contract.[12] The schools' choice function includes a quasi-contract if one of the contracts in the quasi-contract is in the choice correspondence. Under this structure, the quasi-contracts satisfy BLS and IRC; therefore, there is a stable allocation of quasi-contracts. This allocation of quasi-contracts corresponds to an allocation of contracts.

[11] Stable matches cannot be found by breaking ties arbitrarily, as blocking coalitions can include students who are indifferent between their contracts under original allocation and the blocking allocation.

[12] We call these quasi-contracts because they depend on the student's preferences.

5 Adaptive Assignment Algorithm

To accommodate schools and students that are indifferent between contracts, we modify the student-optimal cumulative offer process. Under the Adaptive Assignment Algorithm, students submit an ordered list of their indifference sets of acceptable contracts, starting with their preferred set of contracts. Each set of contracts must contain only one school but can contain multiple contracts with different statuses at that school. Students that prefer one admission status to another can list the same school multiple times in multiple sets of contracts. Given their ranking of contracts, students are considered for schools sequentially based on their preference list. Under truthful reporting, the preference list of student i, $\mathcal{Y}_i = (Y_i^1, Y_i^2, \ldots)$, is found by letting $Y_i^1 = \mathcal{C}_i(X_i)$ and iteratively setting $Y_i^l = \mathcal{C}_i(X_i' \setminus \bigcup_{j=1}^{l-1} Y_i^j)$.

Each school submits a priority list that provides a ranking of individual students. The school's ranking of students induces a choice correspondence $\mathcal{C}_s$ using Algorithm 1. Given the set of available students, the algorithm chooses the highest-ranked set of students that satisfies the quota. Although the algorithm only requires schools state their priority order over individual students, the choice correspondence chooses a unique set of students without making subjective judgements about the schools' rankings over sets of students.

Students start by applying to their preferred school with their preferred set of contracts. Given the available contracts, the Adaptive Assignment Algorithm uses the schools' choice correspondences to choose the preferred set of students to conditionally admit to the schools. The schools retain all the contracts in the indifference sets of conditionally accepted students, allowing these contracts to be used as the algorithm proceeds. Students whose contracts are not in the school's choice correspondence are rejected, and they apply to another school with their next highest set of contracts. The process repeats until every rejected student has no remaining contracts in their preference list. The algorithm is well-defined, finite, and assigns each student to at most one school. The Adaptive Assignment Algorithm is formalized in Algorithm 2.

Algorithm 2. *Students list preferences over contracts:* $\{\mathcal{Y}_i\}_{i \in I}$

(i) The students offer $Y = \bigcup_{i \in I} Y_i^1$

(ii) Student i *is conditionally accepted when* $i \in \boldsymbol{i}(\mathcal{C}_s(Y))$

(iii) Student i *is rejected when* $i \notin \boldsymbol{i}(\mathcal{C}_s(Y))$

(iv) Keep the contracts Y_i^l *in* Y *if student* i *is conditionally accepted*

(v) Otherwise reject Y_i^l *and add* Y_i^{l+1} *to* Y *and go to step (ii)*

The algorithm terminates when there are no new contracts added to Y*.*

Given a final set of contracts Y', the students assigned to school s are $\mathbf{i}(Y'_s)$. The algorithm can choose any $X' \subseteq Y'$ such that $X'_s \in \mathcal{C}_s(Y)$ for all s, as the students and schools are indifferent between any of these allocations. Based on their stated preferences, students are indifferent between the contracts in each of their indifferent sets. As each student's contracts in Y' consist of at most one indifference set, students are indifferent between any of the allocations. Schools are indifferent between any of these allocations, as each allocation in their choice correspondence has the same set of students.

The Adaptive Assignment Algorithm has similar properties to the student-proposing cumulative offer process in the standard school choice environment. The algorithm produces the student-optimal stable allocation (Gale and Shapley, 1962) and provides students the incentive to truthfully report their preferences (Roth, 1982; Dubins and Freedman, 1981). However, schools have similar strategic incentives as under the student-proposing cumulative offer process. Schools may have incentives to misreport their preferences over students (Roth, 1982) or reduce their quotas (Sönmez, 1997).

Proposition 9.

(i) Given truthful reporting of preferences, the Adaptive Assignment Algorithm chooses the student-optimal stable allocation.

(ii) The Adaptive Assignment Algorithm is strategy-proof for the students.

Under truthful reporting, the Adaptive Assignment Algorithm chooses an allocation that is IR for the students, as students do not list schools that are unacceptable. IR is satisfied for the schools because schools do have unacceptable students in their choice correspondence. Finally, any potential blocking coalition that leaves the school better off includes a student whose contract is not considered in the Adaptive Assignment Algorithm. However, that student is worse off under that contract than she is under the Adaptive Assignment Algorithm allocation. Therefore, there is no blocking coalition and the allocation is stable.

The Adaptive Assignment Algorithm allocation is student-optimal because any allocation X' that is preferred to the Adaptive Assignment Algorithm allocation by some student has a contract that is rejected at some stage of the algorithm. In the earliest stage that a student is rejected, there is a school s where a contract in X' is rejected and the set of students Y' that apply to school s. The students rejected by school s are not $\mathcal{C}_s(Y')$ so they are not in $\mathcal{C}_s(Y' \cup X')$. Therefore, any $X'' \in \mathcal{C}_s(Y' \cup X')$ is preferred to X' by s. Further, every student in $\mathbf{i}(X'')$ prefers their allocation under X'' to their allocation under X' as the contracts in X' have not be rejected in a previous stage of the algorithm. Therefore, X'' is a blocking allocation and X' is not stable.

The proof that the Adaptive Assignment Algorithm is strategy-proof follows a similar approach to Dubins and Freedman (1981); however, the method in Dubins and Freedman (1981) cannot be directly applied when preference lists consist of set of contracts. Nonetheless, there is an equivalency between allocations that can be realized when a student states a preference list consisting of sets of contracts and a preference list consisting of single contracts. This equivalency allows us to state the Scenario Lemma for when the misreporting student has a preference list consisting of single contracts.[13] Using the lemma, we apply the rest of the proof of Dubins and Freedman (1981) to show that our mechanism is strategy-proof.

Just as in the cumulative offer process for the standard school choice environment, the Adaptive Assignment Algorithm respects improvements (Balinski and Sönmez, 1999). An improvement for student i occurs when each school's ranking of students i improves and when each schools relative ranking of every other student remains the same. A mechanism respects improvement when improvements never lead to worse allocations for student receiving the improvement. Formally,

Definition 7. *A ranking of students* $\bar{\Pi}$ *represents an* ***improvement*** *over* Π *for student* i *if for all* $s \in S$

1. $i \pi_s i'$ *implies* $i \bar{\pi}_s i'$ *for all* $i' \in I$*, and*

2. $i' \pi_s i''$ *if and only if* $i' \bar{\pi}_s i''$ *for all* $i', i'' \in I \backslash \{i\}$.

A mechanism ***respects improvements*** *if, for all* $i \in I$*, the allocation student* i *receives under the improvement is weakly preferred to the original allocation.*

Proposition 10. *The Adaptive Assignment Algorithm respects improvements.*

The Adaptive Assignment Algorithm respects improvements for the same reason that the deferred acceptance respects improvements in the unconstrained school choice environment and resembles the argument used in Aygun and Turhan (2018). Given an allocation under the Adaptive Assignment Algorithm, a student can achieve the same allocation as under their original allocation by choosing to report only the contract that she is accepted under. If the student reports the same contract under the improvement, then she is accepted under that contract, as her ranking has improved. The allocation under the improvement with truthful reporting is weakly preferred to the allocation where the student reports the single contract, as the Adaptive Assignment Algorithm is strategy-proof. Therefore, her allocation under the improvement is weakly preferred to her original allocation.

6 Conclusion

We introduce a framework for analyzing a school choice problem where there are group specific quotas. Students can simultaneously belong to multiple groups, and each students can only be admitted under a

[13]The students that are truthfully reporting maintain preference lists that consist of sets of contracts.

single status. Students have weak preferences over their admittance status at a school and strict preferences over allocations that include different schools. Schools have responsive preferences over students based on a priority ranking of individual students. Given their priority ranking, schools choose the highest-ranked students possible given the constraints imposed by their quotas. As many allocations fill the same number of privileged seats and have the same students, schools have weak preferences over allocations.

We develop a cumulative offer process where students submit rankings over sets of contracts, allowing students to apply to a school by stating the sets of statuses that they are indifferent between. A school that finds any contract in a students' indifference set acceptable holds all the student's contracts, giving it the flexibility to reassign the student to a different status as the algorithm proceeds and new students' contracts become available to the school. The school rejects all of the student's contracts when none of the student's contracts are in the school's choice correspondence, as these contracts will never be acceptable. The algorithm proceeds with rejected students applying to the next school in their preference list, and terminates when there are no new contracts to consider. The mechanism produces the student-optimal stable allocation. It is strategy-proof and respects improvements.

We build on the previous literature, generalizing the overlapping quota model of Hatfield and Milgrom (2005) to allow students to have weak preferences over their admittance statuses. Additionally, our Adaptive Assignment Algorithm provides a computationally efficient alternative to the method suggested in Hatfield and Milgrom (2005) and extended in Ágoston et al. (2018). To determine the school's choice set, Hatfield and Milgrom (2005) and Ágoston et al. (2018) consider all possible combinations of allocations of students to schools and statuses. The computational time of determining the school's choice correspondence is exponential in the number of students admitted to a school. Alternatively, the processes in the Adaptive Assignment Algorithm are linear in the number of students. The minimum admissible student is the lowest-ranked student in the binding set, and the binding set can be determined quickly by maintaining a tally of the number of students contained in each set of statuses.[14]

We generalize the students' preferences from Aygun and Turhan (2018), allowing students to be indifferent to their admittance statuses. When students have weak preferences over their admittance status, applying their method using arbitrarily tie-breaking violates stability. Specifically, higher-ranked students that are admitted in our environment can be rejected in favor of lower-ranked students in Aygun and Turhan (2018). However, our environment does not have as flexible a reassignment procedure for unused seats. When students have strict preferences over their admittance status, our model is equivalent to Aygun and Turhan (2018) under a specific monotone capacity transfer. Hard bounds is equivalent to the monotone

[14]Under soft bounds, the minimum admissible student is found by choosing the lowest-ranked student in the span of the student's contracts.

capacity transfer where empty seats are not transferred to other statuses, whereas soft bounds is equivalent to transferring unused seats to the general status.[15] Therefore, when selecting between the models, there is a tradeoff between stability when students are indifferent between statuses and a more robust quota reassignment procedure for unused seats.

Our mechanism differs from alternative mechanisms for matching with constraints. It differs from slot specific priorities (Kominers and Sönmez, 2016), where there are multiple seat types each with a different priority rankings over students. The priority rankings for a given seat type depends on the students' ranking and whether they hold the status associated with the seat. In our framework, seats do not have individual priority rankings over students. When a student is added to the school's set of students, the statuses of other students adjust to accommodate students that are added. The addition of a student of one status can cause a student of another status to be rejected, even though the first student is not eligible for the second student's seat. Therefore, the acceptability of a student for a specific status depends on the priority ranking of students applying to other statuses.

The methods in other constrained matching frameworks cannot be applied to our environment. In our framework, constraints are based on the student's statuses, so our framework differs from models that include regional quotas (Hatfield et al., 2017; Kamada and Kojima, 2015, 2017). The presence of overlapping quotas and indifference prevents constrained matching frameworks with type-specific constraints (Kojima et al., 2018; Fragiadakis and Troyan, 2017; Goto et al., 2017) from being applied to our environment. Although the method from Kamada and Kojima (2017) uses a weaker stability notion, the decreasing quotas structure and the binding set accomplish similar objectives. They limit the types of seats that students can apply to, taking into account the set of seats that a student can fill when other students are reallocated to different statuses.

Our approach differs from other models of matching with indifference. In our framework, students are indifferent between statuses at the same school, whereas the literature focuses on schools that are indifferent between students (Erdil and Ergin, 2008; Abdulkadiroğlu et al., 2009). The typical approach of breaking ties randomly and using a cumulative offer process to generate stable allocations cannot be applied to our setting. Randomly matching can assign a student to the wrong status, causing higher-ranked students to be rejected in favor of lower-ranked students, thereby violating stability. Further, improvement cycles resembling the stable improvement cycles used in Erdil and Ergin (2008) do not eliminate blocking coalitions. For example, a student that is assigned to her top-ranked school solely due to random tie breaking is never part of an improvement cycle. Even after the improvement cycles occur, her allocation may continue to have a blocking

[15] The theory behind soft bounds is developed in Appendix C. The soft bounds reassignment procedure can be modified to satisfy other objectives, such as giving other privileged students priority over unused seats.

coalition.

There are limitations in matching with disjoint quotas that were pointed out in other work. As matching with disjoint quotas is a particular case of matching with overlapping quotas, these limitations apply to our work. In particular, quotas can make some minority students worse off (Doğan, 2016). The most systematic problems are eliminated by using minority reserves rather than majority quotas; however, it is still possible for some students to be worse off. Further, our mechanism does not maximize diversity across all schools (Bo, 2016). Instead, our structure gives privileged students priority until the quota threshold is met.

Finally, to implement our mechanism, the procedure for eliciting preferences should be structured to encourage students truthfully state their preferences when they are indifferent between contracts. The mechanism should make easy for students to state indifference, the such as allowing an option of applying to the school under all available statuses. Further, the market designer needs to communicate that students are not harmed by stating their true preferences. Because our mechanism is strategy-proof, students do not receive a worse allocation from stating their true preferences.

7 Acknowledgements

We wish to thank seminar participants at Boğaziçi University, Istanbul Technical University, and Middle East Technical University for the comments and suggestions. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Appendix A Proofs

Prop. 1. **(⇒)** We use induction on the element of T'. Let Z' be feasible for $T' = \{t'\}$. Define $\mathbf{i}_{\subseteq}(Y, T') = \{i' \in \mathbf{i}(Y) : \mathbf{t}(Y) \subseteq T'\}$. Then $X' = \{(i, s, t') : \mathbf{t}(Z'_i) \in T'\}$ satisfies $\mathbf{i}(X') = \mathbf{i}_{\subseteq}(Z'_{i'}, T')$.

Let (⇒) hold for all $T' \subseteq T$ such that Z' is feasible for T' and $|T'| \leq N$. Let T' such that Z' is feasible for T' and $|T'| = N+1$. If there is a $T'' \subset T'$ such that Z' is full for T'', by the inductive step there is an X'' with $\mathbf{i}(X'') = \mathbf{i}_{\subseteq}(Z'_i, T'')$. Define $T''' = T' \setminus T''$. We show $Z'' = \{x \in Z' : \mathbf{t}(x) \nsubseteq T''\}$ is feasible for T'''. Let $T^{(4)} \subseteq T'''$. Then $T^{(4)} \cup T'' \subseteq T'$ so Z' is feasible for $T^{(4)} \cup T'$. Further, $x' \in Z''$ *iff* $x' \notin \{x \in Z' : Z'_i \subseteq T''\}$; therefore,

$$
\begin{aligned}
|\mathbf{i}_{\subseteq}(Z'', T^{(4)})| \leq& |\mathbf{i}_{\subseteq}(Z', T^{(4)} \cup T'')| - |\mathbf{i}_{\subseteq}(Z', T'')| \\
\leq& \sum_{t \in T'} \bar{q}_t^s - \sum_{t \in T''} \bar{q}_t^s = \sum_{t \in T'^{(4)}} \bar{q}_t^s
\end{aligned}
$$

So Z'' is feasible for T''' and $|T'''| \leq N$; so, there is an X''' such that $\mathbf{i}(X''') = \mathbf{i}_{\subseteq}(Z'', T^{(4)})$. Define $X' = X'' \cup X'''$. If $i' \in \mathbf{i}(X')$, then $i' \in \mathbf{i}(X'')$ or $i' \in \mathbf{i}(X''')$. In either case $i' \in \mathbf{i}_{\subseteq}(Z', T')$. Alternatively, let $i' \in \mathbf{i}_{\subseteq}(Z', T')$. If $\mathbf{t}(Z'_i) \subseteq T''$ then $i' \in \mathbf{i}(X'') \subseteq \mathbf{i}(X')$. If $\mathbf{t}(Z'_{i'}) \nsubseteq T''$ then $i \in \mathbf{i}(X''') \subseteq \mathbf{i}(X')$. So, $\mathbf{i}(X') = \mathbf{i}_{\subseteq}(Z', T')$.

If no $T'' \subset T'$ is full for Z'. Order $I' = \mathbf{i}_{\subseteq}(Z', T')$ by $I' = \{i_1, \dots, i_{|I'|}\}$. Let $Z^0 = \bigcup_{\mathbf{t}(Z_i) \subseteq T'} Z'_i$, define Z^j iteratively: for $j = 1, \dots, |I'|$ choose an $x \in Z'_{i_j,s}$ and set $Z^j = (Z^{j-1} \setminus Z'_{i_j}) \cup x$. As Z' is not full for any $T'' \subset T'$, for $j = 0$, $|\mathbf{i}_{\subseteq}(Z', T'')| < \sum_{t \in T'} \bar{q}^s_t$ for all $T'' \subset T'$. If $|\mathbf{i}_{\subseteq}(Z^j, T'')| < \sum_{t \in T''} \bar{q}^s_t$ for all j and all $T'' \subset T'$, then $X' = Z^{|I'|}$ has $|X'| = 1$ for $i' \in I'$, and satisfies the $|X'_t| < \bar{q}^s_t$ for all $t \in T$; therefore, X' is an allocation. By construction, $\mathbf{i}(X') = \mathbf{i}_{\subseteq}(Z', T')$. If $|\mathbf{i}_{\subseteq}(Z^j, T'')| \geq \sum_{t \in T''} \bar{q}^s_t$ for some j and $T'' \subset T'$, there is a minimum j^* and an associated $T''' \subset T'$ where $|\mathbf{i}_{\subseteq}(Z^{j^*}, T''')| \geq \sum_{t \in T'''} \bar{q}^s_t$. Comparing $j^* - 1$ to j^*, the only i for which $\mathbf{t}(Z^j_i)$ changes is i_{j^*}; therefore, $|\mathbf{i}_{\subseteq}(Z', T''')| = \sum_{t \in T'''} \bar{q}^s_t$ and $T''' \subset T'$ is full. By the early part of the proof, there is an X' such that $\mathbf{i}(X') = \mathbf{i}_{\subseteq}(Z', T')$.

(⇐) Let $T' \subseteq T$. Assume $X' \subseteq Z'$ satisfies $\mathbf{i}(X') = \mathbf{i}_{\subseteq}(Z', T')$. For any $T'' \subseteq T'$, $\sum_{t \in T''} X'_t \leq \sum_{t \in T''} \bar{q}^s_t$. Further, $\mathbf{t}(Z'_i) \subseteq T''$ implies $\mathbf{t}(X'_i) \subseteq T''$; therefore, $|\mathbf{i}_{\subseteq}(Z', T'')| \leq |X'_{T''}| \leq \sum_{t \in T''} \bar{q}^s_t$. So, Z' is feasible for T'.

Prop. 2. **(i)** **(⇒)** Using the contrapositive, let $\mathbf{t}(B(Y_{i,s}, Z')) = T'$. Then $|\mathbf{i}_{\subseteq}(Z', T')| = \sum_{t \in T'} \bar{q}^s_t$. Because $\mathbf{t}(Y_{i,s}) \subseteq T'$, $|\mathbf{i}_{\subseteq}(Z'', T')| = |\mathbf{i}_{\subseteq}(Z', T')| + 1 > \sum_{t \in T'} \bar{q}^s_t$ and $Y_{i,s} \cup Z'$ is not feasible. **(⇐)** Let $B(Y_{i,s}, Z') = \infty$. Define $Z'' = Z' \cup Y_{i,s}$ and $\mathbf{i}_{\subseteq}(Y, T') = \{i' \in \mathbf{i}(Y) : \mathbf{t}(Y) \subseteq T'\}$. Since Z' is feasible, $|\mathbf{i}_{\subseteq}(Z', T')| \leq \sum_{t \in T'} \bar{q}^s_t$ for all $T' \subseteq T$. Further, for any T' where $\mathbf{t}(Y_{i,s}) \nsubseteq T'$, $\mathbf{i}_{\subseteq}(Z'', T') = \mathbf{i}_{\subseteq}(Z', T')$; therefore, $|\mathbf{i}_{\subseteq}(Z'', T')| \leq \sum_{t \in T'} \bar{q}^s_t$. As $B(Y_{i,s}, Z') = \infty$, $|\mathbf{i}_{\subseteq}(Z', T')| < \sum_{t \in T'} \bar{q}^s_t$ when $\mathbf{t}(Y_{i,s}) \subseteq T'$. As $\mathbf{i}_{\subseteq}(Z'', T') \subseteq \mathbf{i}_{\subseteq}(Z', T') \cup \{i\}$, $|\mathbf{i}_{\subseteq}(Z'', T')| \subseteq |\mathbf{i}_{\subseteq}(Z', T')| + 1 \leq \sum_{t \in T'} \bar{q}^s_t$. Therefore, Z'' is feasible.

(ii) Define $Z'' = Z' \cup Y_{i,s} \setminus Z'_{i'}$ and $T' = \mathbf{t}(B(Y_{i,s}, Z'))$. **(⇒)** Let Z'' be feasible. As Z' is full for T', $|\mathbf{i}_{\subseteq}(Z', T')| = \sum_{t \in T'} \bar{q}^s_t$. Assume $i' \notin B(Y_{i,s}, Z')$ then $\mathbf{i}_{\subseteq}(Z'', T') = \mathbf{i}_{\subseteq}(Z', T') \cup \{i\}$. Therefore, $|\mathbf{i}_{\subseteq}(Z'', T')| = |\mathbf{i}_{\subseteq}(Z', T')| + 1 > \sum_{t \in T'} \bar{q}^s_t$, contradicting the feasibility of Z''.

(⇐) Let $i' \in \mathbf{i}(B(Y_{i,s}))$. Then $|\mathbf{i}_{\subseteq}(Z'', T'')| \leq |\mathbf{i}_{\subseteq}(Z', T'')| + 1$ as $\mathbf{i}_{\subseteq}(Z'', T'') \setminus \mathbf{i}_{\subseteq}(Z', T'') \subseteq \{i'\}$. Any T'' with $|\mathbf{i}_{\subseteq}(Z', T'')| < \sum_{t \in T''} \bar{q}^s_t$ satisfies $|\mathbf{i}_{\subseteq}(Z'', T'')| \leq |\mathbf{i}_{\subseteq}(Z', T'') + 1| \leq \sum_{t \in T''} \bar{q}^s_t$. Therefore, we consider T'' where Z' is full for T'' where (a) $T' \subseteq T''$, (b) $\mathbf{t}(Y_{i,s}) \nsubseteq T''$, and (c) $\mathbf{t}(Y_{i,s}) \subseteq T''$ and $T' \nsubseteq T''$.

(a) If $T' \subseteq T''$ then $i \notin \mathbf{i}_{\subseteq}(Z', T'')$, $i \in \mathbf{i}_{\subseteq}(Z'', T'')$, $i' \in \mathbf{i}_{\subseteq}(Z', T'')$, and $i' \notin \mathbf{i}_{\subseteq}(Z'', T'')$; therefore, $|\mathbf{i}_{\subseteq}(Z'', T'')| = |\mathbf{i}_{\subseteq}(Z', T'') \cup \{i\} \setminus \{i'\}| = |\mathbf{i}_{\subseteq}(Z', T'')|$. So $|\mathbf{i}_{\subseteq}(Z'', T'')| = |\mathbf{i}_{\subseteq}(Z', T'')| = \sum_{t \in T''} \bar{q}^s_t$. **(b)** If $\mathbf{t}(Y_{i,s}) \nsubseteq T''$ then $i \notin \mathbf{i}_{\subseteq}(Z', T'')$ and $i, i' \notin \mathbf{i}_{\subseteq}(Z'', T'')$. So, $|\mathbf{i}_{\subseteq}(Z'', T'')| = |\mathbf{i}_{\subseteq}(Z', T'') \cup \{i\} \setminus \{i'\}| \leq |\mathbf{i}_{\subseteq}(Z', T'')| = \sum_{t \in T''} \bar{q}^s_t$. **(c)** If $\mathbf{t}(Y_{i,s}) \subseteq T''$ and $T' \nsubseteq T''$ then either T'' is binding or some $T''' \subset T''$ is binding, contradicting the uniqueness of $B(Y_{i,s}, Z')$ (Prop. 11).

Prop. 3. Assume Z' is feasible. When $i^* = \emptyset$, $Z'' = Z' \cup Y_{i,s}$ and is feasible; therefore, by Prop. 2 there is an $X'' \subseteq Z''$ with $\mathbf{i}(X'') = \mathbf{i}(Z'')$. For any $X' \in Z'$ with $\mathbf{i}(X') \neq \mathbf{i}(X'')$, $\mathbf{i}(X') \subset \mathbf{i}(X')$. As $i' \pi_s \emptyset$ for all $i' \in Z''$ and preferences are responsive, students in $\mathbf{i}(X'') \setminus \mathbf{i}(X')$ can be iteratively added to $X^1 = \{x \in X'' : \mathbf{i}(x') \in \mathbf{i}(X')\}$ to form a sequence $X^1, \ldots, X^L = X''$ such that $X^1 \sim_s X''$ and $X^l \succsim_s X^{l-1}$ for $l \in \{2, ..., L\}$; therefore, $X'' = X^l \succsim_s X'$.

When $i^* \pi_s \emptyset$, by Prop. 2, there exists an allocation $X'' \in Z''$. Further, by Prop. 2, any allocation X' satisfies $\mathbf{i}(X') \subseteq \mathbf{i}(Z') \cup \{i\} \setminus \{i'\}$ for some $i' \in B(Y_{i,s}, Z') \cup \{i\}$. If $\mathbf{i}(X') \neq \mathbf{i}(Z'')$ then **(a)** $\mathbf{i}(X') \subseteq \mathbf{i}(X'')$ or **(b)** $\mathbf{i}(X') \subseteq \mathbf{i}(Z') \cup \{i\} \setminus \{i'\}$ for some $i' \in B(Y_{i,s}, Z')$. If (a) then as $\mathbf{i}(X'') \subset \mathbf{i}(X')$, $X'' \succ_s X'$. If (b) then there exists some X''' such that $X' \subseteq X'''$ and $\mathbf{i}(X''') = \mathbf{i}(Z') \cup \{i\} \setminus \{i'\}$ for some $i' \in B(Y_{i,s}, Z')$; therefore, $X''' \succsim_s X'$. Then, as $\mathbf{i}(X'') = \mathbf{i}(X') \cup \{i^*\} \setminus \{i'\}$. By the definition of i^*, $i' \pi_s i^*$; therefore, $X'' \succ_s X''' \succsim_s X'$, as preferences are responsive.

Prop. 4. **(⇒)** Let $X' \in \mathcal{C}_s(Y)$. Take $Z' = \{x \in Y_s : \mathbf{i}(x) \in \mathbf{i}(X')\}$. **(i)** By Prop. 1, Z' is feasible. **(ii)** Assume to the contrary that $\emptyset \, \pi_s \, i'$ for some $i' \in \mathbf{i}(X')$. As $\succsim_s$ is responsive, $X'' = X' \setminus X'_{i'} \succ_s X'$, contradicting $X' \in \mathcal{C}_s(Y)$. **(iii)** If $i \pi_s i^*$ for some $i \in \mathbf{i}(Y_s) \setminus (Z')$, then by Prop. 3 there is an $X'' \subseteq Z' \cup Y_i \setminus Z'_{i^*,s}$ such that $X'' \succ_s X'$. Contradicting $X' \in \mathcal{C}_s(Y)$.

(⇐) Assume (i), (ii), and (iii) are satisfied for some Z'. As $\succsim_s$ is a preorder, there is a maximal element X' of Y_s under $\succsim_s$. The proof of (⇒) shows that Z' with $\mathbf{i}(Z') = \mathbf{i}(X')$ satisfy (i), (ii), and (iii). We show that any Z' that satisfy the conditions have the same students. Assume Z' and Z'' satisfy (i), (ii), and (iii), but $\mathbf{i}(Z') \neq \mathbf{i}(Z'')$. Then there is a maximally ranked student $i \in (\mathbf{i}(Z') \setminus (Z'')) \cup (\mathbf{i}(Z'') \setminus (Z'))$. WLOG let $i \in \mathbf{i}(Z') \setminus \mathbf{i}(Z'')$. Let $i^* = i^*(Z'_i, Z'')$. Either $B(Z'_i, Z'') = \infty$ and $i^* = \emptyset$ or there is an $i' \in \mathbf{i}(B(Z'_i, Z''))$ such that $i' \notin \mathbf{i}(Z')$ and $i \pi_s i'$. If $i^* = \emptyset$ then $i \pi_s i^*$ by (iii); otherwise, $i \pi_s i' R^s i^*$. Both cases contradict (iii). Therefore, $\mathbf{i}(Z') = \mathbf{i}(Z'')$. For any $X'' \in Z'$ with $\mathbf{i}(X'') = \mathbf{i}(Z')$, $\mathbf{i}(X'') = \mathbf{i}(X')$ and $X' \sim_s X''$; therefore, $X' \succsim X'''$ for all $X''' \in Y_s$, so $X' \in \mathcal{C}_s(Y)$.

Prop. 5. We show that Z' satisfies (i), (ii), and (iii) of Prop. 4. **(i)** $Z^0 = \emptyset$ is feasible. So if Z^{l-1} is feasible, then by Prop. 3, Z^l is feasible. By induction, $Z' = Z^{|\mathbf{i}(Y_s)|}$ **(ii)** We show $i \pi_s \emptyset$ for all $i \in \mathbf{i}(Z')$. This is trivially true for $Z^0 = \emptyset$. Assume $i \pi_s \emptyset$ for all $i \in \mathbf{i}(Z^{l-1})$. Then $i^* = i^*(Y_{i_l,s}, Z^{l-1}) \succsim \emptyset$ as $i^* = \emptyset$ or $i^* \in \mathbf{i}(Z^{l-1})$. So, if $i_j \in \mathbf{i}(Z^l)$ then $i_j \pi_s i^* \succsim \emptyset$. By induction, $i \pi_s \emptyset$ for all $i \in Z'$ **(iii)** We show that $i^*_l = i^*(Y_{i_k,s}, Z^l)$ is increasing in l. By (ii), $i^*_l R_s \emptyset$ for all l. If $i^*_l = \emptyset$ then $i^*_{l+1} R^s i^*_l$, so assume $i^*_l \neq \emptyset$. Define $T_l = \mathbf{t}(B(Y_{i_k,s}, Z^l))$ and $I_l = \mathbf{i}(B(Y_{i_k,s}, Z^l))$. If $T_{l+1} \subseteq T_l$, then either $i^*(Y_{i_l}, Z^l) \pi_s i_{l+1}$ or $i_l \pi_s i^*(Y_{i_l}, Z^l)$. When $i^*(Y_{i_l}, Z^l) \pi_s i_l$, $I_l = I_{l+1}$, and $i^*_{l+1} = i^*_{l+1}$. Otherwise, $I_{l+1} \subseteq I_l \cup \{i_l\}$ and $i^*_{l+1} R^s i^*_{l+1}$.

If $T_{l+1} \nsubseteq T_l$, then $i^*(Y_{i_l,s}, Z^l) \in I_l$. Let $B_{k,l} = B(Y_{i_k,s}, Z^l) \cup B(Y_{i_l,s}, Z^l))$. As $\mathbf{t}(B_{k,l})$ is the union of two full sets, it is full. Because $B(Y_{i_l,s}, Z^l) \subseteq B_{k,l}$, $\mathbf{t}(Y_{i_l}) \in \mathbf{t}(B_{k,l})$ and $i_l, i^*(Y_{i_l,s}, Z^l) \in B_{k,l}$; therefore, $B_{k,l}$ is full for Z^{l+1}. As $\mathbf{t}(Y_{i_k}) \subseteq \mathbf{t}(B_{k,l})$, $I_{l+1} \subseteq \mathbf{i}(B_{k,l}) \cup \{i_l\}$. As $i^*(Y_{i_l,s}, Z^l) \in I_l$, $i^*(Y_{i_l,s}, Z^l) R^s i_l^*$. Therefore, $i' R^s i_l^*$ for $i' \in B_{k,l} \cup i_l$. As, $B(Y_{i_k,s}, Z^{l+1}) \subseteq B_{k,l} \cup \{Y_{i_l}\}$ the result follows.

Prop. 6. Let $Y \subset X$, $x \in X \setminus Y$, and $x \notin X'$ for all $X' \in \mathcal{C}_s(Y \cup \{x\})$. Let $X'' \in \mathcal{C}_s(Y)$, then $X'' \succsim_s X'''$ for all $X''' \in Y_s$. As $x \notin X'$ for any $X' \in \mathcal{C}_s$, there exist an X^4 with $x \notin X^4$ such that $X^4 \in Y_s \cup \{x\}$. As $x \notin X^4$, $X^4 \in Y_s$. Since $X'' \in \mathcal{C}_s(Y)$ and $X'' \succsim_s X^4$, $X'' \in \mathcal{C}_s(Y \cup \{x\})$. Alternatively, let $X'' \in \mathcal{C}_s(Y \cup \{x\})$, $X'' \succsim_s X'''$ for all $X''' \in Y \cup \{x\}$; therefore, $X'' \succsim_s X'''$ for all $X''' \in Y$. Since $x \notin X''$, $X'' \in Y$ and $X'' \in \mathcal{C}_s(Y)$. Therefore, $\mathcal{C}_s(Y) = \mathcal{C}_s(Y \cup \{x\})$.

Prop. 7. **(Substitutability, ULS)** Example 6 show that $\mathcal{C}_s$ violates substitutability and ULS. **(BLS)** Let $Y' = Y \cup \{x, x'\}$ and take any sequence of students $\{i_1, \ldots, i_{|\mathbf{i}(Y)|}, \mathbf{i}(x), \mathbf{i}(x')\}$ and run Algorithm 1. If $x \notin \mathcal{C}_s(Y \cup \{x\})$ then $x \notin Z^{|\mathbf{i}(Y)|+1}$; therefore, $x \notin Z^{|\mathbf{i}(Y)|+2}$, and $x \notin \mathcal{C}_s(Y \cup \{x, x'\})$.

Prop. 8. For $(X, \{\succsim_i\}_{i\in I}, \{\succsim_s\}_{s\in S})$ let $\bar{t}(x) = \{\mathbf{t}(x) : x' \sim_i x\}$ and $\bar{X} = \{(i, s, \bar{t}) : x \in X\}$. For $\bar{x} \in \bar{X}$, let $\phi(\bar{x}) = \{x \in X : \mathbf{i}(x) = \mathbf{i}(\bar{x}), \mathbf{s}(x) = \mathbf{s}(\bar{x}), \mathbf{t}(x) \in \mathbf{t}(\bar{x})\}$ and for $\bar{Y} \subseteq \bar{X}$, let $\Phi(\bar{Y}) = \{\phi(\bar{x}) : \bar{x} \in \bar{Y}\}$. Let $\bar{C}_i(\bar{Y}) = \phi^{-1}(\mathcal{C}_i(\Phi(\bar{Y}))$ and $\bar{\mathcal{C}}_s^0(\bar{Y}) = \{\Psi(X') : X' \in \mathcal{C}_s(\Phi(\bar{Y}))\}$ where $\Psi(X') = \{\bar{x} \in \bar{X} : X' \cap \phi(\bar{x}) \neq \emptyset\}$. For $l \in \{1, \ldots, |I|\}$ define $\bar{\mathcal{C}}_s^l(\bar{Y}) = \{\bar{X}' \in \bar{\mathcal{C}}_s^{l-1}(\bar{Y}) : \bar{X}'_{i_k} = \bar{C}_i(\bigcup \bar{X})' \in \bar{\mathcal{C}}_s^{l-1}(\bar{Y})\}$ and let $\bar{C}_s(\bar{Y}) = \bar{\mathcal{C}}_s^{|I|}(\bar{Y})$. Then $\bar{C}_s(\bar{Y})$ is well-defined and consists of a single allocation. **(IRC)** take $\bar{x} \in \bar{X} \setminus \bar{Y}$ where $\bar{x} \notin \bar{C}_s(\bar{Y} \cup \{\bar{x}\})$. If $\bar{x}' \notin \bar{X}'$ for any $\bar{X}' \in \bar{\mathcal{C}}_s^0(\bar{Y} \cup \{\bar{x}\})$, then $\bar{\mathcal{C}}_s^0(\bar{Y} \cup \{\bar{x}\}) = \bar{\mathcal{C}}_s^0(\bar{Y})$. If $\bar{x}' \in \bar{X}'$ for some $\bar{X}' \in \bar{\mathcal{C}}_s^0(\bar{Y} \cup \{\bar{x}\})$, no $\bar{X}'$ with $\bar{x} \in \bar{X}'$ is not in $\bar{C}_i(\bar{\mathcal{C}}_s^{l-1}(\bar{Y}))$ for some l, otherwise $\bar{x} \in \bar{C}_s(\bar{Y} \cup \bar{x})$. Therefore, $\bar{C}_s$ satisfies IRC. **(BLS)** If $\bar{x} \notin \bar{C}_s(\bar{Y} \cup \{\bar{x}\})$, then $\bar{x} \notin \bar{X}'$ for some $X' \in \bar{\mathcal{C}}_s^0(\bar{Y} \cup \{\bar{x}\})$. As $\mathbf{i}(\bar{X}') = \mathbf{i}(\bar{X}'')$ for all $X', X'' \in \mathcal{C}_s(\Phi(\bar{Y} \cup \{\bar{x}\}))$, $\mathbf{i}(x) \notin \bar{X}'$ for any $X' \in \mathcal{C}_s(\Phi(\bar{Y} \cup \{\bar{x}\}))$. So, using Algorithm 1, $\mathbf{i}(x) \notin \bar{X}'$ for any $X' \in \mathcal{C}_s(\Phi(\bar{Y} \cup \{\bar{x}\} \cup \{\bar{x}'\}))$. Therefore, $\bar{x} \notin \bar{C}_s(\bar{Y} \cup \{\bar{x}\} \cup \{\bar{x}'\})$. Using the results from Hatfield and Kojima (2010) and Aygün et al. (2012) there is a stable allocation in $\bar{X}' \in \bar{X}$. Any allocation of all the students in $X' \in \Phi(\bar{X}')$ to seat at the school is stable, as any X'_i that violates IR corresponds would cause $\bar{X}'_i$ to violate IR and any X'' that violates UB would correspond to an $\bar{X}''$ that violates UB.

Prop. 9. **(i, Stability)** Let Y' be feasible set when the Adaptive Assignment Algorithm allocation (AAA) terminates with an allocation $X' \subseteq Y'$ such that $\mathbf{i}(X'_s) = \mathbf{i}(Y's)$ for all $s \in S$. As $X'_i \in Y_i^l$ for some l, $X_i \succsim_i \emptyset$; therefore, $X'_i \in \mathcal{C}_i(X')$ and X' is IR for the students. As $X'_s \in \mathcal{C}_s(Y')$, $i\pi_s\emptyset$ for all $i \in \mathbf{i}(X'_s)$; therefore, by responsively, $X'_s \succsim_s X''_s$ for any $X'' \subset X'$; so, $X'_s \in \mathcal{C}_s(X')$ and X' is IR for the schools. Let $X'' \in \mathcal{C}_s(X' \cup X'')$ such that $X'' \succ_s X'$, then $X'' \notin \mathcal{C}_s(Y)$; otherwise, $X'' \sim_s X'$. Therefore, there exists an $i \in \mathbf{i}(X'')$ such that X''_i was not considered by the AAA; therefore, $X'_i \succ X''_i$. Therefore, X' is UB. **(Student-optimal)** Assume X' is the AAA allocation, and let $X'' \succ_i X'$ for some $i \in I$. Then there is a stage in the AAA where X''_i is rejected. WLOG, take i to be a student that is rejected at the earliest stage that a contract in X'' is rejected. Then, letting $s = \mathbf{s}(X''_i)$ and Y' be the contracts at the start of the earliest rejection stage. Here $i \notin \mathbf{i}(\mathcal{C}_s(Y'))$, so $i \notin \mathbf{i}(\mathcal{C}_s(Y \cup X''))$. Take any $X''' \in \mathcal{C}_s(Y' \cup X'')$, then $X''' \succ_s X''$ and $X''' \in \mathcal{C}_s(X''' \cup X'')$. Further, $X'''_{i'} \succsim_{i'} X''_{i'}$ for any $i' \in \mathbf{i}(X''')$ as no contracts in X'' were rejected in any previous stage of the AAA. Therefore, X'' is not stable.

(ii) To show that AAA is strategy-proof, we show that the scenario lemma from Dubins and Freedman (1981) holds in our environment. Then the proof in Dubins and Freedman (1981) applies verbatim. We consider whether i' benefits from misreporting her preferences. We represent the preference lists of student i by $\mathcal{Y}_i = (Y_i^1, \dots, Y_i^{l_i})$, all students by $\mathcal{Y} = \{\mathcal{Y}_i\}_{i \in I}$, and all students but i' by $\mathcal{Y}_{-i'} = \{\mathcal{Y}_i\}_{i \in I \setminus \{i'\}}$.

Let X' be an AAA allocation given the stated preferences $\mathcal{Y}$. We construct a preference list for i' consisting of single contracts that gives X' as an AAA allocation. If $X'_{i'} = \emptyset$, let $\beta = \bigcup_{\forall j} \{Y_i^j\}$ and choose an enumeration $\bar{\mathcal{Y}}_i = (\{x_1\}, \dots, \{x_{|\beta_i|}\})$ with $x_j \in \beta$. If $X'_{i'} \neq \emptyset$, then $X'_{i'} \in Y_{i'}^{\alpha}$ for some $Y_{i'}^{\alpha} \in \mathcal{Y}_{i'}$. Let $\beta = \{Y_i^j : j < \alpha\}$ and choose a $\bar{\mathcal{Y}}_i = (\{x_1\}, \dots, \{x_{|\beta_i|}\}, X'_i)$ with $x_j \in \beta$. X' is stable for the preferences implied by $\bar{\mathcal{Y}} = \bar{\mathcal{Y}}_{i'} \cup \mathcal{Y}_{-i'}$ as any blocking coalition under $\bar{\mathcal{Y}}$ is a blocking coalition under $\mathcal{Y}$. Further, any stable allocation under the preferences implied by $\bar{\mathcal{Y}}$ is stable under the preferences implied by $\mathcal{Y}$. As the AAA allocation is student-optimal under $\mathcal{Y}$, the allocation X' is student-optimal under $\bar{\mathcal{Y}}$. Therefore, X' is an AAA allocation under $\bar{\mathcal{Y}}$. Thus, i' can profitably misreport preferences if and only if there is a beneficial misrepresentation of the form $\bar{\mathcal{Y}}$, allowing us to restrict our attention to misrepresentations of the form $\bar{\mathcal{Y}}$.

We prove the scenario lemma: Suppose that every application in $\bar{\mathcal{Y}}_i$ occurs in $\bar{\mathcal{Y}}$. Then for any $\bar{\mathcal{Y}}'_i$ with $\{Y_i^{\alpha} : Y_i^{\alpha} \in \bar{\mathcal{Y}}'_i\} \subseteq \{Y_i^{\alpha} : Y_i^{\alpha} \in \bar{\mathcal{Y}}_i\}$ the applications and rejections occurring in $\bar{\mathcal{Y}}'$ also occur in $\bar{\mathcal{Y}}$.

As the AAA outcome is independent of the order that student preference list are considered, WLOG we assume that all students other than i' are processed by the algorithm first. Student i' applies to the first school on her preference list. The AAA proceeds sequentially, with the rejected student applying to the next school in her preference list. The algorithm terminates when either a student is accepted without causing another student to be rejected or a student that is rejected does not have any other contracts in her preference list. We use induction on the number of rounds of rejections and new applications that occur after the first application of i' under $\bar{\mathcal{Y}}'$. When zero rounds have occurred, student i' has not applied to any schools, so $\bar{\mathcal{Y}}'$ and $\bar{\mathcal{Y}}$ give the same rejections. Student i' applies using a contract in $\bar{\mathcal{Y}}'_i$ which, by assumption, also occurs in $\bar{\mathcal{Y}}$.

Assume that every rejection and associated new application that have appeared in the previous k rounds of $\bar{\mathcal{Y}}'$ have also occurred at some point in the AAA under $\bar{\mathcal{Y}}$. Consider a student $i \in I$ that applies to school s in round k under contracts Y_i^m that causes a student i'' to be rejected from s in round $k+1$. Let $i^* = i^*(Y_i^m, Y_s^k)$ where Y_s^k is the set of contracts allocated to the school at the end of round k. The lowest-ranked student of i and i^* is rejected. By the inductive assumption, all the students in $i \cup B(Y_i^m, Y_s^k)$

have applied to s; therefore, the minimum student of i and i^* has been rejected under $\bar{\mathcal{Y}}$. Let i'' be the student rejected from s. If $i'' = i'$, her next application occurs as all of the application in $\bar{\mathcal{Y}}'_i$ are in $\bar{\mathcal{Y}}_i$ by assumption. If the student is $i \neq i'$, then she has been rejected from s under both $\bar{\mathcal{Y}}'$ and $\bar{\mathcal{Y}}$, so she applies to the same school under $\bar{\mathcal{Y}}$ as under $\bar{\mathcal{Y}}'$. Thus, by induction, the scenario lemma holds.

Using the scenario lemma, the remainder of the proof in Dubins and Freedman (1981) applies verbatim, showing that the AAA is strategy-proof.

Prop. 10. Let X' be the allocation under the AAA given the preference profile Π, and let $\bar{\Pi}$ be an improvement for i. When student i states a preference list only including the contract $\{X'_i\}$, the final allocation assigns student i the contract $\{X'_i\}$. Under the improvement, the ranking of student i is higher than it is under π, so i is preferred to the minimal admissible student. The same steps of rejections occur under $\bar{\Pi}$ as under Π. When student i' is considered for school $\mathbf{s}(X'_i)$, she is accepted under Π; therefore, since her priority ranking is higher under $\bar{\Pi}$ she is accepted under $\bar{\Pi}$ too. As the other students relative rankings remains the same, the same series of rejections occur under both priority rules. Student i is never rejected under Π; therefore, she is not rejected under $\bar{\Pi}$. Since the mechanism is strategy-proof, the allocation under truthful reporting under $\bar{\Pi}$ is weakly preferred to reporting $\{X'_i\}$, so it is weakly preferred to the allocation under Π.

Appendix B Technical Appendix

Proposition 11. *Assume Z' is feasible for T^p. Let $Y'_s \subseteq X'_s$ with $\boldsymbol{i}(Y'_s) \cap \boldsymbol{i}(Z) = \emptyset$, then $B(Y'_s, Z')$ is well-defined.*

Prop. 11. If there is no T' such that Z' is full for T' and $\mathbf{t}(Y_s) \subseteq T'$, then $B(Y'_s, Z') = \emptyset$. So, let T' and T'' be full sets of Z' that contain $\mathbf{t}(Y'_s)$. We show that $T_1 \cap T_2$ is also full and contains $\mathbf{t}(Y'_s)$, so $T_1 = T_2$. Define $\tilde{T}' = T' \setminus (T' \cap T'')$, $q^s_{T' \cap T''} = |\{i \in \mathbf{i}(Z') : \mathbf{t}(Z'_i) \subseteq T'''\}|$, and $q^s_{T'''} = |\{i \in \mathbf{i}(Z') : \mathbf{t}(Z'_i) \subseteq T''', \mathbf{t}(Z'_i) \nsubseteq T' \cap T''\}|$ for any $T''' \subseteq \{T', T''\}$. As Z' is feasible,

$$q^s_{\tilde{T}'} + q^s_{T' \cap T''} + q^s_{\tilde{T}''} \leq \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t + \sum_{t \in \tilde{T}''} \bar{q}^s_t$$

Subtracting $q^s_{T'''} + q^s_{T' \cap T''} = \sum_{t \in T'''} \bar{q}^s_t + \sum_{t \in T' \cap T''} q^s_t$ for $T''' \in \{\tilde{T}', \tilde{T}''\}$ from both sides gives $-q^s_{T' \cap T''} \leq -\sum_{t \in T' \cap T''} \bar{q}^s_t$ or $q_{T' \cap T''} \geq \sum_{t \in T' \cap T''} \bar{q}^s_t$. Further, $\mathbf{t}(Y'_s) \subseteq T'$ and $\mathbf{t}(Y'_s) \subseteq T''$, $\mathbf{t}(Y'_s) \subseteq T' \cap T''$, so the binding set is unique, and hence well-defined.

Corollary 2. *Assume Z' is full for T' and T''. Then $T' \cap T''$ and $T' \cup T''$ are full for Z'.*

Corr. 2. Proposition 11 shows that $T''' = T' \cap T''$ is full for Z'. Using the notation from Proposition 11, we have

$$q^s_{\tilde{T}'} + q^s_{T' \cap T''} = \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t$$

as T', T'' and T''' are full. Therefore, $q^s_{\tilde{T}'} = \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t - q^s_{T' \cap T''}$. Similarly, $q^s_{\tilde{T}''} = \sum_{t \in \tilde{T}''} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t - q^s_{T' \cap T''}$

Defining $q^s_{T' \cup T''} = |\{i \in \mathbf{i}(Z') : \mathbf{t}(Z'_i) \subseteq T' \cup T''\}|$

$$q^s_{T' \cup T''} \geq q^s_{\tilde{T}'} + q^s_{T' \cap T''} + q^s_{\tilde{T}''} = \sum_{t \in \tilde{T}'} \bar{q}^s_t + \sum_{t \in T' \cap T''} \bar{q}^s_t + \sum_{t \in \tilde{T}''} \bar{q}^s_t$$

So $T' \cup T''$ is full.

Appendix C Soft Bounds

We extend the model to soft bounds. Given this structure, we characterize feasibility, replaceability, and the choice correspondence accounting for the soft bounds.

Appendix C.1 Model

Under soft bounds, seats that are not allocated to a privileged student are available for general students. Therefore, an allocation is an $X' \subseteq X$ such that $|X'_i| \leq 1$ for all $i \in I$, $|X'_s| \leq \bar{q}^s$ for all $s \in S$, and $|X'_{s,t}| \leq \bar{q}^s_t$ for all $s \in S$ and $t \in T^p$.

The schools primary requirement is to choose an allocation that maximizes the number of acceptable students allocated to privileged seats. When comparing two allocations that fill the same number of privileged seats, the school's preferences are responsive with respect to π_s. Defining $q^s_p(X') = |\{x \in X'_{T^p} : \mathbf{i}(x)\pi_s\emptyset\}|$, an allocation $X' \succsim_s X''$ if either (i) $q^s_p(X') > q^s_p(X'')$, or (ii) $q^s_p(X') = q^s_p(X'')$ and $\mathbf{i}(X') \succsim^r_s \mathbf{i}(X'')$, where $\succsim^r_s$ are the responsive preferences over students induced by π_s. The school's preferences induce a preorder $\succsim_s$ on $\mathcal{X}_s$.

To determine the school's choice correspondence under soft bounds, we consider a set of contracts Z' that is partitioned into students assigned to privileged seats and students assigned to general seats. Given a set of available contracts Y, the school chooses a set of students $I' \subseteq \mathbf{i}(Y_s)$ with corresponding contracts $Z' = \{x \in Y_s : \mathbf{i}(x) \in I'\}$. The students are partitioned into students assigned to privileged seats, $\tilde{I}'$, and students assigned to general seats, $\hat{I}'$. The available contracts for the students assigned to privileged and general seats are $\tilde{Z}' = \{x \in Y_s : \mathbf{i}(x) \in \tilde{I}'\}$ and $\hat{Z}' = \{x \in Y_s : \mathbf{i}(x) \in \hat{I}'\}$, respectively. For students in Z' to be allocated to their designated seats, we require $\mathbf{t}(\tilde{Z}'_i) \cap T^p \neq \emptyset$ for all $i \in \tilde{I}'$ and $t_0 \in \mathbf{t}(\hat{Z}'_i)$ for all $i \in \hat{I}'$.

Appendix C.2 Feasibility

For soft bounds, feasibility is defined over sets of privileged students, $\tilde{Z}'$. It is also defined for partitions, Z', and accounts for the soft bounds by allowing any unfilled privileged seats to be available to general students. Feasibility for soft bounds is similar to feasibility for hard bounds. However, students in $\tilde{Z}'$ are only allocated to privileged seats; therefore, we ignore their general statuses when considering the feasibility for $T \subseteq T^p$.

Definition 8. *Let $Z' \subseteq X_s$ and $T' \subseteq T^p$. A set of contracts Z' is*

(i) ***feasible*** *for T' if $|\{i \in \boldsymbol{i}(\tilde{Z}') : \boldsymbol{t}(\tilde{Z}'_i) \cap T^p \subseteq T''\}| \leq \sum_{t \in T''} \bar{q}^s_t$ for all $T'' \subseteq T'$.*

(ii) ***full*** *for T' if Z' is feasible for T' and $|\{i \in \boldsymbol{i}(\tilde{Z}') : \boldsymbol{t}(\tilde{Z}'_i) \cap T^p \subseteq T'\}| = \sum_{t \in T'} \bar{q}^s_t$*

A set of contracts Z' is feasible if $\tilde{Z}'$ is feasible for T^p and $|\boldsymbol{i}(Z')| \leq \bar{q}^s$.

Just as for hard bounds, feasibility is linked to the existence of allocations under the school's available contracts. A set of contracts $\tilde{Z}'$ is feasible for T' if and only if the students in $\tilde{Z}'$ whose privileged contracts are contained in T' can be allocated to the privileged seats. A partition Z' is feasible if and only if there is an allocation where students in $\tilde{Z}'$ and $\hat{Z}'$ are allocated to privileged and general seats, respectively.

Proposition 12. *Let $Z' \subset X_x$.*

(i) $\tilde{Z}'$ is feasible for $T' \subseteq T^p$ iff there is an allocation $X' \subseteq \tilde{Z}'_{T^p}$ where $\boldsymbol{i}(X') = \{i \in \boldsymbol{i}(\tilde{Z}') : \boldsymbol{t}(\tilde{Z}'_i) \cap T^p \subseteq T'\}$.

(ii) $Z' \subseteq Y$ is feasible iff there is an allocation $X' \subseteq Z'$ such that $\boldsymbol{i}(X'_{T^p}) = \boldsymbol{i}(\tilde{Z}')$ and $\boldsymbol{i}(X') = \boldsymbol{i}(Z')$.

name=Proof,style=mystyle. **(i,⇒)** Assume $\tilde{Z}'$ is feasible for T', then $\tilde{Z}'' = \tilde{Z}'_{T^p}$ is feasible for T' under hard bounds. By Prop. 1 there is an $X' \subseteq \tilde{Z}'' = \tilde{Z}'_{T^p}$ satisfying $\mathbf{i}(X') = \{i \in \mathbf{i}(\tilde{Z}') : \mathbf{t}(\tilde{Z}'_{i,T^p}) \subseteq T'\} = \{i \in \mathbf{i}(\tilde{Z}') : \mathbf{t}(\tilde{Z}'_i) \cap T^p \subseteq T'\}$. **(i,⇐)** Assume $\mathbf{i}(X') = \{i \in \mathbf{i}(\tilde{Z}') : \mathbf{t}(\tilde{Z}'_{i,T^p}) \subseteq T'\}$. By Prop 1, $\tilde{Z}'' = \tilde{Z}'_{T^p}$ is feasible for T' under hard bounds; therefore, $\tilde{Z}'$ is feasible for T'. **(ii,⇒)** Let Z' be feasible. Then $\tilde{Z}'$ is feasible for T^p. By (i) there is an $X'' \in \tilde{Z}'$ with $\mathbf{i}(X'') = \mathbf{i}(\tilde{Z}')$. As X'' is an allocation, $|X''_t| \leq \bar{q}^s_t$ for all $t \in T^p$. Let $X''' = \hat{Z}'_{t_0}$. As $t_0 \in \hat{Z}'_{i',s}$ for all $i' \in \mathbf{i}(\hat{Z}')$, $\mathbf{i}(X''') = \mathbf{i}(\hat{Z}')$. Let $X' = X'' \cup X'''$. Then $X'' \subseteq \tilde{Z}' \subseteq Z'$ and $X''' \subseteq \hat{Z}' \subseteq Z'$; therefore, $X' \subseteq Z'$. Further, $\mathbf{i}(X') = \mathbf{i}(X'') \cup \mathbf{i}(X''') = \mathbf{i}(\tilde{Z}') \cup \mathbf{i}(\hat{Z}') = \mathbf{i}(Z')$. Therefore, $|X'| = |\mathbf{i}(X')| = |\mathbf{i}(Z')| \leq \bar{q}^s$. **(ii,⇐)** Assume $X' \subseteq Z'$ is an allocation such that $\mathbf{i}(X') = \mathbf{i}(Z')$ and $\mathbf{i}(X_{T^p}) = \mathbf{i}(\tilde{Z}')$. As X' is an allocation $|\mathbf{i}(Z')| = |\mathbf{i}(X')| = |X'| \leq \bar{q}^s$. Further, by (i, ⇒), $\tilde{Z}'$ is feasible for T^p. Therefore, Z' is feasible.

We define the binding set for privileged seats considering that students in $\tilde{Z}'$ are allocated to privileged seats. Just as in the definition of feasibility, we disregard the general contracts of students in $\tilde{Z}'$.

Definition 9. *Assume $\tilde{Z}' \subseteq X_s$ is feasible for T^p. A set of statuses $T' \subseteq T^p$ is binding for the contracts $Y' \subseteq X_s$ under $\tilde{Z}'$ if*

(i) $\boldsymbol{t}(Y'_{T^p}) \subseteq T'$

(ii) $\tilde{Z}'$ *is full for* T'

(iii) There is no T'' such that $\boldsymbol{t}(Y'_{T^p}) \subseteq T'' \subset T'$ and $\tilde{Z}'$ is full for T''

If there is no T' that is binding, we say that the binding set is empty.

The binding contracts, $B(Y', \tilde{Z}')$, are defined similarly to under hard bounds. If a binding set T' exists, then $B(Y', \tilde{Z}') = \bigcup\{\tilde{Z}'_i : \mathbf{t}(\tilde{Z}'_i) \cap T^p \subseteq T'\}$. If Y' does not contain any privileged contacts, then $B(Y', \tilde{Z}') = \emptyset$. If there is no T' such that $\tilde{Z}'$ is full for T' with $\mathbf{t}(Y' \cap T^p) \subseteq T'$, then $B(Y', \tilde{Z}') = \infty$. The binding students and binding statuses are $\mathbf{i}(B(Y', \tilde{Z}'))$ and $\mathbf{t}(B(Y', \tilde{Z}'))$, respectively.

We allow Y' to have multiple students, as it is useful for determining whether a partition maximizes the number of privileged students and whether a general student can replace a privileged student. When Y' has multiple students, the process of replacing students in $\tilde{Z}'$ is similar to when Y' has a single student. When $B(Y', \tilde{Z}') = \infty$, it is possible to add some student from Y' to Z' and maintain feasibility. When $B(Y', \tilde{Z}') \neq \infty$, each student in the binding set can be replaced by some student in $\mathbf{i}(Y')$. Formally,

Proposition 13. *Let $\tilde{Z}' \subset X_s$ be feasible. Let $Y' \subseteq X_s$ and $\boldsymbol{i}(Y') \cap \boldsymbol{i}(\tilde{Z}') = \emptyset$.*

(i) $B(Y', \tilde{Z}') = \infty$ if and only if $\tilde{Z}' \cup Y'_i$ is feasible for T^p for some $i \in \boldsymbol{i}(Y')$.

(ii) If $B(Y', \tilde{Z}') \neq \infty$, then for any $i' \in \boldsymbol{i}(B(Y', \tilde{Z}'))$, $\tilde{Z}' \cup Y'_i \setminus Z'_{i',s}$ is feasible for T^p for some $i \in \boldsymbol{i}(Y')$.

name=Proof,style=mystyle. **(i,⇒)** Let $B(Y', \tilde{Z}') = \infty$. Then no full set T' contains $\mathbf{t}(Y'_{T^p})$. As the union of full sets is full (Corr. 2), there is a $t \in \mathbf{t}(Y'_{T^p})$ with $t \notin T'$ for any full $T' \subseteq T^p$. Take an $x \in Y'$ with $\mathbf{t}(x) = t$ and let $\tilde{Z}'' = \tilde{Z}' \cup Y_i$. Define $\mathbf{i}_{\subseteq}(Y, T') = \{i' \in \mathbf{i}(Y) : \mathbf{t}(Y_{T^p}) \subseteq T'\}$. Then $\mathbf{i}_{\subseteq}(\tilde{Z}'', T') = \mathbf{i}_{\subseteq}(\tilde{Z}', T') \leq \sum_{t \in T'} \bar{q}^s_t$ for any T' where $t \notin T'$. If $t \in T'$, then $\mathbf{i}_{\subseteq}(\tilde{Z}', T') < \sum_{t \in T'} \bar{q}^s_t$; therefore, $\mathbf{i}_{\subseteq}(\tilde{Z}'', T') \leq \mathbf{i}_{\subseteq}(\tilde{Z}', T') + 1 \leq \sum_{t \in T'} \bar{q}^s_t$. So $\tilde{Z}''$ is feasible for T^p. **(i,⇐)** Let $Z'' = \tilde{Z}' \cup Y_i$ be feasible. Then $\mathbf{i}_{\subseteq}(\tilde{Z}'', T') < \mathbf{i}_{\subseteq}(\tilde{Z}', T') \leq \sum_{t \in T'} \bar{q}^s_t$ for any T' with $\mathbf{t}(Y') \subseteq T'$; therefore, T' is not full. Hence, $B(Y', \tilde{Z}') = \infty$.
(ii) Let $B(Y', \tilde{Z}') \neq \infty$ and take $i' \in \mathbf{i}(B(Y', \tilde{Z}'))$. Let $Z'' = Z' \setminus Z'_{i'}$. By the proof of (i), $B(Y', \tilde{Z}'') = \infty$. So, by (i) there is an $i \in Y'_i$ such that $Z'' \cup Y'_i = Z' \cup Y'_i \setminus Z'_{i'}$ is feasible.

We impose additional structure on the partition. A partition Z' is valid when it is feasible and there is no student in $\mathbf{i}(\hat{Z}')$ whose contracts can be added to $\tilde{Z}'$ while maintaining feasibility; therefore, Z' is valid if it is impossible to fill more privileged seats using the students in $\hat{Z}'$.

Definition 10. *A partition $Z' = \tilde{Z}' \cup \hat{Z}'$ is **valid**, if Z' is feasible and there is no $i \in \hat{I}'$ such that $\tilde{Z}' \cup \hat{Z}'_i$ is feasible for T^p.*

When a Z' is valid, it is impossible to add a student from $\hat{Z}'$ to $\tilde{Z}'$; therefore, $B(\hat{Z}', \tilde{Z}') \neq \infty$. Alternatively, if $B(\hat{Z}', \tilde{Z}') \neq \infty$, then $\hat{Z}'$ is contained in a full set. So, any $\hat{Z}'_i$ is also contained in a full set and cannot be added to $\tilde{Z}'$ with replacing another student. This proves the following corollary to Prop. 13.

Corollary 3. *A feasible partition Z' is valid if and only if $B(\hat{Z}', \tilde{Z}') \neq \infty$.*

We analyze valid partitions because they fill the maximum number of privileged seats possible using the contracts in Z'; therefore, schools always prefer a valid partition to other partitions containing the same contracts. Further, valid partitions with the same set of students fill the same number of privileged seats; therefore, the school is indifferent between them. These results, stated formally below, allow us to restrict our attention to valid sets of contracts.

Proposition 14. *Let $Z' \subseteq X_s$ be feasible. Then*

(i) There exists a valid partition of $\tilde{Z}'' \cup \hat{Z}'' = \tilde{Z}' \cup \hat{Z}'$ such that $|\boldsymbol{i}(\tilde{Z}'')| \geq |\boldsymbol{i}(\tilde{Z}')|$.

(ii) Any valid partitions Z' and Z'' with $\tilde{Z}'' \cup \hat{Z}'' = \tilde{Z}' \cup \hat{Z}'$ have $|\boldsymbol{i}(\tilde{Z}')| = |\boldsymbol{i}(\tilde{Z}'')|$.

name=Proof,style=mystyle. **(i)** Let $\tilde{Z}^0 = \tilde{Z}'$ and $I' = \mathbf{i}(\hat{Z}') = \{i_1, \ldots, i_{|I'|}\}$. For $l \in \{1, \ldots, |I'|\}$, define $\tilde{Z}^l = \tilde{Z}^{l-1} \cup \hat{Z}'_{i_l}$ if $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) = \infty$ and $\tilde{Z}^l = \tilde{Z}^{l-1}$ if $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) \neq \infty$. Let $\tilde{Z}'' = \tilde{Z}^{|I'|}$ and $\hat{Z}'' = \hat{Z}' \setminus \tilde{Z}'$.

$\tilde{Z}^0 = \tilde{Z}'$ is feasible for T^p. Assume $\tilde{Z}^{l-1}$ is feasible for T^p. Define $\mathbf{i}_{\subseteq}(Y, T') = |\{i \in \mathbf{i}(Y) : \mathbf{t}(Y_{i,T^p}) \subseteq T'\}|$. If $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) \neq \infty$ then $\tilde{Z}^l = \tilde{Z}^{l-1}$ is feasible for T^p. If $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) = \infty$, then for any $T' \nsubseteq \mathbf{t}(\hat{Z}'_{i_l,T^p})$, $\mathbf{i}_{\subseteq}(\tilde{Z}^l, T') = \mathbf{i}_{\subseteq}(\tilde{Z}^{l-1}, T') \leq \sum_{t \in T'} \bar{q}^s_t$. For any $T' \subseteq \mathbf{t}(\hat{Z}'_{i_l,T^p})$, $\mathbf{i}_{\subseteq}(\tilde{Z}^{l-1}, T') < \sum_{t \in T'} \bar{q}^s_t$; therefore, $\mathbf{i}_{\subseteq}(\tilde{Z}^l, T') = \mathbf{i}_{\subseteq}(\tilde{Z}^{l-1}, T') + 1 \leq \sum_{t \in T'} \bar{q}^s_t$. So, $\tilde{Z}^l$ is feasible for T^p. As Z' is feasible, $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| \leq \bar{q}^s$; therefore, Z'' is feasible.

We show that Z'' is valid. Take $i_l \in \hat{Z}''$, then $B(\hat{Z}'_{i_l}, \tilde{Z}^{l-1}) \neq \infty$. Let $T_l \subseteq T^p = \bigcup\{T' : \mathbf{t}(\hat{Z}'_{l,T^p}) \subseteq T' \subseteq T^p$ and T' is full for $Z^l\}$. Then $T_{l-1} \subseteq T_l$, so $B(\hat{Z}'_{i_l}, \tilde{Z}') \neq \infty$ and Z'' is valid.

(ii) Let Z' and Z'' be valid with $\tilde{Z}' \cup \hat{Z}' = \tilde{Z}'' \cup \hat{Z}''$. We show that $|\mathbf{i}(\tilde{Z}')| = |\mathbf{i}(\tilde{Z}'')|$. Let $T' \subseteq T^p$ and $T'' \subseteq T^p$ be the union of the full sets in $\tilde{Z}'$ and $\tilde{Z}''$, respectively. By Corr. 2, T' and T'' are full under Z' and Z'', respectively. If $T' \neq T''$, WLOG let $T''' = T' \setminus T'' \neq \emptyset$. As T' is full under Z', $\tilde{I}' = \mathbf{i}_{\subseteq}(\tilde{Z}', T') \setminus \mathbf{i}_{\subseteq}(\tilde{Z}', T'')$ satisfies $|\tilde{I}'| \geq \sum_{t \in T'''} \bar{q}^s_t$. As T' is not full under Z'', $\tilde{I}'' = \mathbf{i}_{\subseteq}(\tilde{Z}'', T') \setminus \mathbf{i}_{\subseteq}(\tilde{Z}'', T'')$ satisfies $|\tilde{I}''| < \sum_{t \in T'''} \bar{q}^s_t$.

Therefore, there is some $i \in \tilde{I}' \setminus \tilde{I}''$. As $\mathbf{t}(Z''_{i,T^p}) \nsubseteq T''$, $B(\hat{Z}''_i, \tilde{Z}'') = \infty$, contradicting the validity of Z''. Therefore, $T' = T''$. Let $\hat{I}' = \mathbf{i}(\tilde{Z}' \setminus \tilde{Z}'_{T'})$ and $\hat{I}'' = \mathbf{i}(\tilde{Z}'' \setminus \tilde{Z}''_{T'})$. If there is an $i \in \hat{I}' \setminus \hat{I}''$, then $B(\tilde{Z}''_i, \tilde{Z}'') = \infty$, contradicting the validity of Z''. Therefore, $\hat{I}' = \hat{I}''$. Since Z' and Z'' are both full under T' and have the same students in $T^p \setminus T'$, $|\mathbf{i}(\tilde{Z}')| = |\mathbf{i}(\tilde{Z}'')|$.

As valid partitions with the same contracts have the same students and fill the same number of privileged seats, schools are indifferent between the allocations associated them. Thus, the preferences over the allocations induce preferences over Z'. Specifically, for valid Z' and Z'' and their associated X' and X'', we say $Z' \succsim_s Z''$ if and only if $X' \succsim_s X''$.

The following lemma is useful.

Lemma 1. *Let Z' be a valid partition such that $i\,\pi_s\,\emptyset$ for all $i \in \boldsymbol{i}(Z')$. Then $Z' \succ_s Z''$ for any valid partition of Z'' where $Z'' = Z' \setminus Z'_{I'}$ for some $I' \subseteq \boldsymbol{i}(Z_i)$.*

name=Proof,style=mystyle. We show the result for $I' = \{i\}$ for some $i \in \mathbf{i}(Z')$. The lemma follows from iteratively removing $i \in I'$ and using transitivity. Let $Z'' = \tilde{Z}'' \cup \hat{Z}''$ be a valid partition. If $B(Z'_i, \tilde{Z}'') = \infty$, then $Z''' = Z'' \cup Z'_i$ is a valid with $\tilde{Z}''' = \tilde{Z}'' \cup Z'_i$ and $\hat{Z}''' = \hat{Z}''$. Therefore, $|\mathbf{i}(\tilde{Z}'')| < |\mathbf{i}(\tilde{Z}''')|$. As Z''' is valid, by Prop 14, $|\mathbf{i}(\tilde{Z}''')| = |\mathbf{i}(\tilde{Z}')|$; therefore, $|\mathbf{i}(\tilde{Z}'')| < |\mathbf{i}(\tilde{Z}')|$ and $Z' \succ_s Z''$. If $B(Z'_i, \tilde{Z}'') \neq \infty$, then $Z''' = Z'' \cup Z'_i$ is a valid with $\tilde{Z}''' = \tilde{Z}''$ and $\hat{Z}''' = \hat{Z}'' \cup Z'_i$. Therefore, $|\mathbf{i}(\tilde{Z}'')| = |\mathbf{i}(\tilde{Z}''')|$. As Z''' is valid, by Prop 14, $|\mathbf{i}(\tilde{Z}''')| = |\mathbf{i}(\tilde{Z}')|$; therefore, $|\mathbf{i}(\tilde{Z}'')| = |\mathbf{i}(\tilde{Z}')|$. As $Z' = Z'' \cup \{i\}$, $i\pi_s\emptyset$, and preferences are responsive, $Z' \succ_s Z''$.

Appendix C.3 Replaceability

We consider whether it is beneficial to add student $i \notin \mathbf{i}(Z')$, with contracts $Y_{i,s}$, to Z', by determining student i can be added to Z' without replacing another student and when it is necessary to replace a student in $\mathbf{i}(Z')$. When it is necessary to replace a student, there are multiple ways student i can replace student $i' \in \mathbf{i}(Z')$. The student can directly replace i' without affecting other students in the partition, or i can indirectly replace i' with some other students transferring between privileged and general seats. The span of a set of contracts determines replaceability.

Definition 11. *Let $Y' \subseteq X_s$ and let $Z' \subseteq X_s$ be valid. Then*

$$Span(Y', Z') = \begin{cases} \hat{B}(Z') \cup B(\hat{Z}' \cup Y', \tilde{Z}') & \text{if } t_0 \in \boldsymbol{t}(Y' \cup B(Y', Z')) \\ B(Y', \tilde{Z}') & \text{otherwise} \end{cases}$$

where $\hat{B}(Z') = \hat{Z}'$ if $|Z'| = \bar{q}^s$ and $\hat{B}(Z') = \infty$ if $|Z'| < \bar{q}^s$.

The $Span(Y_{i,s}, Z')$ is the set of contracts that student i can replace. It allows for direct replacement, when a student replaces a student in the binding set, and indirect replacement, through chains of replacement across partitions. For instance, student i can replace a student in $B(Y_{i,s}, \tilde{Z}')$, who can replace a student in the general partition, who in turn can replace a privileged student i' who is not in $B(Y_{i,s}, \tilde{Z}')$. The minimum admissible student is the lowest-ranked student whose contracts are contained in $Span(Y', Z')$.

Definition 12. *Let $Y' \subseteq X_s$ and let $Z' \subseteq X_s$ be valid. The **minimum admissible student** is*

$$i^*(Y', Z') = \begin{cases} \min_{\pi_s} \boldsymbol{i}(Span(Y', Z')) & \text{if } Span(Y', Z') \neq \infty \\ \emptyset & \text{if } Span(Y', Z') = \infty \end{cases}$$

The minimum admissible student helps to establish the optimal replacement procedure. When i is acceptable and $i^*(Y_{i,s}, Z') = \emptyset$, adding student i fills more privileged seats or lead to higher-ranked students. Therefore, student i should be added. When $|\mathbf{i}(Z')| < \bar{q}^s$, student i can be added without removing another student. However, when $|\mathbf{i}(Z')| = \bar{q}^s$, it is necessary to remove a student from Z'. Removing the minimum ranked student in the span of $\hat{Z}'$ is optimal. Formally,

Proposition 15. *Assume $Z' \subseteq X_s$ is valid, $Y_{i,s} \neq \emptyset$ for some $i \notin \boldsymbol{i}(\tilde{Z}')$, $i \, \pi_s \, \emptyset$ for all $i \in \boldsymbol{i}(Z' \cup Y_{i,s})$, and $i^*(Y_{i,s}, Z') = \emptyset$.*

(i) If $|Z'| < \bar{q}^s$ then there exists a valid $Z'' = Z' \cup Y_{i,s}$.

(ii) If $|Z'| = \bar{q}^s$ then there exists a valid $Z'' = Z' \cup Y_{i,s} \setminus Z'_{i^*}$ where $i^* = i^*(\hat{Z}', Z')$.

Further, $Z'' \succ_s Z'''$ for any valid $Z''' \subseteq Z' \cup Y_{i,s}$ where $\boldsymbol{i}(Z''') \neq \boldsymbol{i}(Z'')$.

name=Proof,style=mystyle. **(i)** If $B(Y_{i,s}, \tilde{Z}') = \infty$, let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}$ and $\hat{Z}'' = \hat{Z}'$. By Prop 13, $\tilde{Z}''$ is feasible for T^p. As $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| + 1 \leq \bar{q}^s$, Z'' is feasible. Z' is valid, so $B(\hat{Z}', \tilde{Z}')) \neq \infty$. $B(\hat{Z}', \tilde{Z}'') = B(\hat{Z}', \tilde{Z}'') \neq \infty$ as $\mathbf{t}(Y_{i,s}) \nsubseteq B(\hat{Z}', \tilde{Z}')$; therefore, Z'' is valid. If $B(Y_{i,s}, \tilde{Z}') \neq \infty$ then there is an $i' \in i \cup \mathbf{i}(B(Y_{i,s}, \tilde{Z}'))$ with $t_0 \in \mathbf{t}(Y_{i',s})$. Take $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s} \setminus Y_{i',s}$ and $\hat{Z}'' = \hat{Z}' \cup Y_{i',s}$. By Prop 13, Z'' is feasible for T^p. Further, $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| + 1 \leq \bar{q}^s$, so Z'' is feasible. As Z' is valid, $\mathbf{t}(B(\hat{Z}', \tilde{Z}')) = T'$ for some full T'. Further, $\mathbf{t}(Y_{i',s}) \cap T^p \subseteq T''$ for some full T''; therefore, $\mathbf{t}(\hat{Z}' \cup Y_{i',s}) \cap T^p \subseteq T' \cup T''$ and Z'' is valid. **(ii)** As $i^*(Y_{i,s}, Z') = \emptyset$ and $|\mathbf{i}(Z')| = \bar{q}^s$, $B(Y_{i,s}, \tilde{Z}') = \infty$. If $i^* \in \mathbf{i}(\hat{Z}')$, let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s}$ and $\hat{Z}'' = \hat{Z}' \setminus \hat{Z}'_{i^*}$. As $B(Y_{i,s}, \tilde{Z}') = \infty$, $\tilde{Z}''$ is feasible for T^p by Prop 13. As $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| = \bar{q}^s$, Z' is feasible. $\mathbf{t}(B(\hat{Z}'', \tilde{Z}'')) \subseteq \mathbf{t}(B(\hat{Z}', \tilde{Z}'))$ as $\hat{Z}'' \subseteq \hat{Z}'$ and $\tilde{Z}' \subseteq \tilde{Z}''$. Therefore, the validity of Z' implies Z'' is valid. If $i^* \in \mathbf{i}(\tilde{Z}')$ then $i^* \in \mathbf{i}(B(\hat{Z}', \tilde{Z}''))$. By Prop. 13, there is an $i' \in \mathbf{i}(\hat{Z}')$ such that $\tilde{Z}''' = \tilde{Z}' \cup \hat{Z}'_{i',s} \setminus \tilde{Z}'_{i^*}$ is feasible for T^p. Further, $B(Y_{i,z}, Z''') = \infty$ so $\tilde{Z}'' = \tilde{Z}' \cup \hat{Z}'_{i',s} \cup Y_{i,s} \setminus \tilde{Z}'_{i^*,s}$ is feasible for T^p. Let $\hat{Z}'' = \hat{Z}' \setminus \hat{Z}'_{i',s}$. Here, $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| = \bar{q}^s$. As Z'

is valid, $B(\hat{Z}', \tilde{Z}') \neq \infty$. As $\hat{Z}'' \subseteq \hat{Z}'$ and $\mathbf{t}(\hat{Z}'_{i',s}) \in B(\hat{Z}', \tilde{Z}')$, $\mathbf{t}(B(\hat{Z}'', \tilde{Z}'')) \subseteq \mathbf{t}(B(\hat{Z}', \tilde{Z}')) \neq \infty$, so Z'' is valid.

($Z'' \succsim_s Z'''$, i) When $\mathbf{i}(Z''') \neq \mathbf{i}(Z'')$, $Z''' \subset Z''$. Therefore, by Lemma 1, $Z'' \succ_s Z'''$.

($Z'' \succsim_s Z'''$, ii) Since $|\mathbf{i}(Z' \cup Y_{i,s})| = \bar{q}^s + 1$, $Z''' \subseteq Z^4 = Z'_i \cup Y_{i,s} \setminus Z''_{i'}$ for some $i' \in \mathbf{i}(Z'')$. WLOG, take Z^4 to be valid. If $i' \in \mathbf{i}(Span(\hat{Z}', Z')$ then $i' \pi_s i^*$. Further, $|\mathbf{i}(\tilde{Z}^4)| \leq |\mathbf{i}(\tilde{Z}'')|$; otherwise, $i^* \notin \mathbf{i}(Span(\hat{Z}', Z')$. Therefore, as $Z'' = Z^4 \cup Z''_{i'} \setminus Z'_{i^*}$ for $i' \pi^s i^*$ and by Lemma 1, $Z'' \succ_s Z^4 \succsim_s Z'''$.

If $i' \notin Span(\hat{Z}', Z')$, then $i' \in B(Y_{i,s}, \tilde{Z}')$ or $i' \notin B(Y_{i,s}, \tilde{Z}')$. If $i' \in B(Y_{i,s}, \tilde{Z}')$ then $i' \pi i^*$; therefore, by the argument above, $Z'' \succ_s Z^4 \succsim_s Z'''$. If $i' \notin B(Y_{i,s}, \tilde{Z}')$ and $i' \notin Span(\hat{Z}', Z')$, then the set of students that are not contained in $T' = \mathbf{t}(B(Y_{i,s}, \tilde{Z}')) \cup \mathbf{t}(Span(\hat{Z}', Z'))$ decreases by 1, and the students with status contained in T' remains the same. Fewer privileged students are allocated under Z^4; therefore, $Z'' \succ_s Z^4 \succsim_s Z'''$.

When $i^* = i^*(Y_{i,s}, Z') \neq \emptyset$, student i cannot be added to the allocation without removing a student from Z'; however, student i can replace the students in $Span(Y_{i,s}, Z')$. Therefore, when $i \succ i^*$, the optimal replacement strategy is for the school to replace the minimum admissible student in $Span(Y_{i,s}, Z')$. The contracts of student i are added to Z' while those of student i^* are removed.

Proposition 16. *Assume that Z' is valid, $Y_{i,s} \neq \emptyset$ for some $i \notin \boldsymbol{i}(\tilde{Z}')$, and $i^* = i^*(Y_{i,s}, Z') \, \pi_s \, \emptyset$. Then $Z'' = Z' \cup Y_{i,s} \setminus Z_{i^*}$ is a valid. Further, if $i \, \pi_s \, i^*$ then $Z'' \succ_s Z'''$ for any valid $Z''' \subseteq Z' \cup Y_{i,s}$ where $Z''' \neq Z''$.*

name=Proof,style=mystyle. **(i)** We show that Z'' is feasible for the following: (a) $i^* \in \mathbf{i}(B(Y_{i,s}), \tilde{Z}')$ (b) $i^* \in \mathbf{i}(\tilde{Z}')$, and (c) $i^* \in \mathbf{i}(B(\hat{Z}', \tilde{Z}') \setminus B(Y_{i,s}, \tilde{Z}'))$. **(a)** If $i^* \in \mathbf{i}(B(Y_{i,s}, \tilde{Z}'))$, $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s} \setminus Y_{i^*,s}$ is feasible for T^p by Prop. 13. As $|\mathbf{i}(\tilde{Z}'')| = |\mathbf{i}(\tilde{Z}'')| \leq \bar{q}^s$, Z'' is feasible. **(b)** If $i^* \in \mathbf{i}(\tilde{Z}')$, there is an $i' \in i \cup \mathbf{i}(B(Y_{i.s}, \tilde{Z}'))$ with $t_0 \in \mathbf{t}(Y_{i',s})$. Let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s} \setminus Y_{i',s}$ and $\hat{Z}'' = \hat{Z}' \cup Y_{i',s} \setminus Y_{i^*,s}$. By Prop. 13, $\tilde{Z}''$ is feasible for T^p. As $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| = \bar{q}^s$, Z'' is feasible. **(c)** When $i^* \in \mathbf{i}(B(\hat{Z}', \tilde{Z}') \setminus B(Y_{i,s}, \tilde{Z}'))$, there is an $i'' \in \hat{Z}'$ such that $\tilde{Z}' \cup \hat{Z}'_{i''} \setminus \tilde{Z}'_{i^*,s}$ is feasible, by Prop. 13. Further, there is an $i' \in i \cup \mathbf{i}(B(Y_{i.s}, \tilde{Z}'))$ with $t_0 \in \mathbf{t}(Y_{i',s})$. Let $\tilde{Z}'' = \tilde{Z}' \cup Y_{i,s} \cup \hat{Z}'_{i''} \setminus Y_{i',s} \setminus \tilde{Z}'_{i^*,s}$ and $\hat{Z}'' = \hat{Z}' \cup Y_{i',s} \setminus Y_{i''}$. As $\tilde{Z}'' = \tilde{Z}''' \cup Y_{i,a} \setminus \tilde{Z}'_{i',s}$, $\tilde{Z}''$ is feasible for T^p by Prop. 13. As $|\mathbf{i}(Z'')| = |\mathbf{i}(Z')| \leq \bar{q}^s$, Z'' is feasible. We show Z'' is valid. Define $\mathbf{i}_{\subseteq}(Y, T') = \{i' \in \mathbf{i}(Y) : \mathbf{t}(Y) \subseteq T' \cup \{t_0\}\}$ and let $T' = \mathbf{t}(Span(Y_{i,s}, Z')) \cap T^p$. Here, $|\mathbf{i}_{\subseteq}(Z', T')| = |\mathbf{i}_{\subseteq}(Z'', T')|$ so Z'' is full under T'. As $\mathbf{t}(\hat{Z}'') \cap T^p \subseteq T'$, $B(\hat{Z}'', \tilde{Z}'') \neq \infty$, so Z'' is valid.

($Z'' \succsim_s Z'''$) $|\tilde{Z}''| \geq |\tilde{Z}'''|$ for all valid $Z''' \subseteq Z' \cup Y_{i,s}$; otherwise, there is an $i \in \mathbf{i}(\tilde{Z}''') \setminus \mathbf{i}(\tilde{Z}'')$ with $B(\tilde{Z}'''_i, \tilde{Z}'') = \infty$, contradicting the validity of Z''. Any $Z''' \subseteq Z' \cup Y_{i,s}$ satisfies $Z''' = Z^4 = Z' \cup Y_{i,s} \setminus Z'_{i'}$ for some $i' \in \mathbf{i}(Span(Y_{i,s}, Z') \cup i$; otherwise, $Span(Y_{i,s}, Z') = \infty$. When $i' \in \mathbf{i}(Span(Y_{i,s}, Z'))$, Z^4 is a substitution of the form (a), (b), or (c). So, when Z''' is valid, $|\mathbf{i}(\tilde{Z}^4)| = |\mathbf{i}(\tilde{Z}')| = |\mathbf{i}(\tilde{Z}'')|$. As preferences are responsive, $Z'' \succ_s Z^4$ for $Z^4 \neq Z''$. By Lemma 1, $Z'' \succ_s Z^4 \succsim_s Z'''$ when $Z^4 \neq Z''$, and $Z'' \sim_s Z^4 \succ_s Z'''$ when $Z''' \subset Z''$. Thus $Z'' \succsim_s Z'''$.

Appendix C.4 Choice Correspondence

Propositions 15 and 16 imply that students that are not in the schools choice correspondence are not preferred to the minimum admissible student. The converse is also true. When we restrict our attention to valid partitions, an allocation is in the school's choice correspondence if and only if all the students in the corresponding Z' are acceptable and there are no students in $Y_s \setminus Z'$ and are preferred to the minimum admissible student.

Proposition 17. *Let $Y \subseteq X$. $X' \in \mathcal{C}_s(Y)$ if and only if the Z' where $\tilde{Z}' = Y_{\boldsymbol{i}(X'_{T^p}),s}$ and $\hat{Z}' = Y_{\boldsymbol{i}(X'_{t_0}),s}$ satisfies the following conditions:*

(i) Validity: Z' is a valid partition

(ii) Acceptability: $i \,\pi_s\, \emptyset$ for all $i \in \boldsymbol{i}(Z')$.

(iii) No beneficial replacement: $i^*(Y_{i,s}, Z') \,\pi_s\, i$ for all $i \in \boldsymbol{i}(Y_s) \setminus \boldsymbol{i}(Z')$

name=Proof,style=mystyle. **(⇒)** Assume $X' \in \mathcal{C}_s(Y)$. **(i)**Then, $\tilde{Z}'$ is feasible for T^p as $X' \setminus X'_{t_0}$ is an allocation. Further, Z' is feasible as $|\mathbf{i}(Z')| = |\mathbf{i}(X')| \leq \bar{q}^s$. Z' is valid; otherwise, there is a Z'' with $|\mathbf{i}(\tilde{Z}'')| > |\mathbf{i}(\tilde{Z}')|$. By Prop. 12, there is an $X'' \subseteq Z''$ with $|X'' \setminus X''_{t_0}| = |\mathbf{i}(\tilde{Z}'')| > |\mathbf{i}(\tilde{Z}')| = |X' \setminus X'_{t_0}|$. Therefore, Z' is valid. **(ii)** If there is an $i' \in \mathbf{i}(X')$ with $\emptyset \,\pi_s\, i'$, then by responsiveness $X'' = X' \setminus X'_{i'} \succ_s X'$, contradicting $X' \in \mathcal{C}_s(Y)$. **(iii)** To the contrary, let $i \,\pi_s\, i^*(Y_{i,s}, Z')$ for some $i \in \mathbf{i}(Y_s \setminus Z')$. Then either $i^*(Y_{i,s}, Z') = \emptyset$ or $i^*(Y_{i,s}, Z') \neq \emptyset$. Therefore, either Prop. 15 or Prop. 16 imply there exists an $Z'' \succ_s Z'$ and an $X'' \subseteq Z''$ satisfying $X'' \succ_s X'$, contradicting $X' \in \mathcal{C}_s(Y)$. Therefore, (iii) holds.

(⇐) Assume (i), (ii), and (iii) are satisfied for some Z'. Then, as $\succsim_s$ is a preorder, there is a maximal element X' under the preorder $\succsim_s$. The proof of (⇒) shows that Z' with $\mathbf{i}(Z') = \mathbf{i}(X')$ satisfy (i), (ii), and (iii). Let Z' and Z'' satisfy (i), (ii), and (iii). Assume that $i(Z') \neq i(Z'')$. Then there is a maximally ranked student in $i \in (\mathbf{i}(Z') \setminus \mathbf{i}(Z'')) \cup (\mathbf{i}(Z'') \setminus \mathbf{i}(Z'))$ under $\succsim_s$. WLOG let $i \in \mathbf{i}(Z') \setminus \mathbf{i}(Z'')$. $Span(Y_{i,s}, Z'') = \infty$ implies $\emptyset \pi_s i$ a contradiction. $Span(Y_{i,s}, Z'') \neq \infty$ implies there is an $i' \in \mathbf{i}(X'') \setminus \mathbf{i}X'$ such that $i' \in Span(Y_{i,s}, Z'')$ so $i \succ_s i' \succsim_s i^*(Y_{i,s}, Z'')$ a contradiction. As $Span(Y_{i,s}, Z'') = \infty$ and $Span(Y_{i,s}, Z'') \neq \infty$ both lead to a contradiction, so $i(Z') = i(Z'')$ and $Z' = Z''$. Therefore, there exists an $X' \in Z'$ with $\mathbf{i}(X') = \mathbf{i}(Z')$ and $X' \in \mathcal{C}_s(Y)$.

The school's choice correspondence can be determined using an iterative process. We consider the school's acceptable contracts on a student-by-student basis. When $i^* = \emptyset$, any acceptable student is conditionally accepted to the school. If the soft-bound seats are full, a student in Z' is removed to accommodate the new student. When $i^* \neq \emptyset$, the student is accepted if she is preferred to the minimum admissible student, and the minimum admissible students is removed.

Algorithm 3. *Let Y with $\boldsymbol{i}(i \in i(Y) : i \succ_s \emptyset) = \{i_j\}_{j \in \{1,\dots,|\boldsymbol{i}(Y)|\}}$. Start with $Z^0 = \emptyset$.*

For each student, i_j, determine $i^* = i^*(Y_{i_j,s}, Z^{j-1})$.

1. *If $i^* = \emptyset$ and $|\boldsymbol{i}(Z^{j-1})| < q^s$, let $Z^j = Z^{j-1} \cup Y_{i_j,s}$*
2. *If $i^* = \emptyset$ and $|\boldsymbol{i}(Z^{j-1})| = q^s$, let $Z^j = Z^{j-1} \cup Y_{i_j,s} \setminus \tilde{Z}^{j-1}_{i'}$ where $i' = i^*(\hat{Z}^{j-1}, Z^{j-1})$*
3. *If $i^* \succsim_s \emptyset$ and $i_j \, \pi_s \, i^*$ where $i^* = i^*(Y_{i_j,s}, Z^{j-1})$, let $Z^j = Z^{j-1} \cup Y_{i,s} \setminus Z^{j-1}_{i^*}$*

The movement of students between $\tilde{Z}'$ and $\hat{Z}'$ is specified in the proofs of Propositions 15 and 16.

In Algorithm 3, the candidate student's contracts are added when it increases the number of privileged seats or improves the overall student ranking. When necessary, an optimal replacement procedure is used to remove a student. The final Z' is preferred because any student not in the allocation was rejected some stage. So the rejected student is worse than the minimum admissible student. The following result shows that the algorithm chooses the schools in the schools choice correspondence.

Proposition 18. *The set of contracts Z' chosen by Algorithm 3 is unique and satisfies $\boldsymbol{i}(\mathcal{C}_s(Y)) = \boldsymbol{i}(Z')$.*

name=Proof,style=mystyle. We show that Z' satisfies (i), (ii), and (iii) of Prop. 17. **(i)** We use induction. $Z^0 = \emptyset$ is feasible. So if Z^{l-1} is feasible, then by Props. 15 and 16, Z^l is feasible. **(ii)** All students in i_j for $j \in \{1, \dots, |\mathbf{i}(Y)|\}$ satisfy $i_j \pi_s \emptyset$ **(iii)** We show that $i^*_l = i^*(Y_{i_k,s}, Z^l)$ is increasing in l. By (ii), $i^*_l \succsim_s \emptyset$ for all l. If $i^*_l = \emptyset$ then $i^*_{l+1} \succsim_s i^*_l$, so assume $i^*_l \neq \emptyset$. Define $T_l = \mathbf{t}(Span(Y_{i_k,s}, Z^l))$ and $I_l = \mathbf{i}(Span(Y_{i_k,s}, Z^l))$. If $T_{l+1} \subseteq T_l$, then $I_{l+1} \subseteq I_l \cup i_{l+1}$. Therefore, for any $i \in I_{l+1}$ either $i = i_{l+1} \succ_s i^*(Y_{i',s}, Z^l) \succsim_s i^*_l$ or $i \succsim_s i^*_l$; so, $i^*_l \succsim_s i^*_l$. If $T_{l+1} \not\subseteq T_l$, then $i^*(Y_{i_l,s}, Z^l) \in I_l$ so $i^*(Y_{i_l,s}, Z^l) \succsim_s i^*(Y_{i_k,s}, Z^l)$. Let $S_{k,l} = Span(Y_{i_k,s}, Z^l) \cup Span(Y_{i_l,s}, Z^l))$. By Prop. 5 $B(Y_{i_k,s}, \tilde{Z}^{l+1}) \subseteq B(Y_{i_k,s}, \tilde{Z}^l) \cup B(Y_{i_l,s}, \tilde{Z}^l)$ and $B(\hat{Z}, \tilde{Z}^{l+1}) \subseteq B(\hat{Z}^l, \tilde{Z}^l)$ if $t_0 \in S_{k,l}$; so, $Span(Y_{i_k,s}, Z^{l+1}) \subseteq S_{k,l}$. As $i^*(Y_{i_l,s}, Z^l) \in I_l$, $i^*(Y_{i_l,s}, Z^l) \succsim_s i^*_l$. Therefore, $i' \succsim_s i^*_l$ for $i' \in S_{k,l} \cup i_l$. As, $Span(Y_{i_k,s}, Z^{l+1}) \subseteq S_{k,l}$ the result follows.

Soft bounds are incorporated into the Adaptive Assignment Algorithm by using Algorithm 3 to determine the school's choice correspondence. The proofs of the results in Section 5 are trivially adapted to incorporate the soft bounds choice correspondence. Therefore, the Adaptive Assignment Algorithm produces the student-optimal stable allocation under soft bounds.

References

Abdulkadiroğlu, A. (2005). College admissions with affirmative action. *International Journal of Game Theory*, 33(4):535–549.

Abdulkadiroğlu, A., Pathak, P. A., and Roth, A. E. (2009). Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match. *American Economic Review*, 99(5):1954–78.

Abdulkadiroğlu, A., Pathak, P. A., Roth, A. E., and Sönmez, T. (2005). The boston public school match. *American Economic Review*, 95(2):368–371.

Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747.

Ágoston, K. C., Biró, P., and Szántó, R. (2018). Stable project allocation under distributional constraints. *Operations Research Perspectives*, 5:59–68.

Aygün, O. and Sönmez, T. (2013). Matching with contracts: Comment. *American Economic Review*, 103(5):2050–51.

Aygün, O., Sönmez, T., et al. (2012). The importance of irrelevance of rejected contracts in matching under weakened substitutes conditions. *Unpublished mimeo.*

Aygün, O. and Turhan, B. (2017). Large-scale affirmative action in school choice: Admissions to IITs in India. *American Economic Review*, 107(5):210–13.

Aygun, O. and Turhan, B. (2018). Dynamic reserves in matching markets. *Unpublished mimeo.*

Balinski, M. and Sönmez, T. (1999). A tale of two mechanisms: student placement. *Journal of Economic theory*, 84(1):73–94.

Bo, I. (2016). Fair implementation of diversity in school choice. *Games and Economic Behavior*, 97:54–63.

Doğan, B. (2016). Responsive affirmative action in school choice. *Journal of Economic Theory*, 165:69–105.

Dubins, L. E. and Freedman, D. A. (1981). Machiavelli and the gale-shapley algorithm. *The American Mathematical Monthly*, 88(7):485–494.

Erdil, A. and Ergin, H. (2008). What's the matter with tie-breaking? improving efficiency in school choice. *American Economic Review*, 98(3):669–89.

Fragiadakis, D. and Troyan, P. (2017). Improving matching under hard distributional constraints. *Theoretical Economics*, 12(2):863–908.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

Goto, M., Kojima, F., Kurata, R., Tamura, A., and Yokoo, M. (2017). Designing matching mechanisms under general distributional constraints. *American Economic Journal: Microeconomics*, 9(2):226–62.

Hafalir, I. E., Yenmez, M. B., and Yildirim, M. A. (2013). Effective affirmative action in school choice. *Theoretical Economics*, 8(2):325–363.

Hatfield, J. W. and Kojima, F. (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory*, 145(5):1704–1723.

Hatfield, J. W., Kominers, S. D., and Westkamp, A. (2017). Stable and strategy-proof matching with flexible allotments. *American Economic Review*, 107(5):214–19.

Hatfield, J. W. and Milgrom, P. R. (2005). Matching with contracts. *American Economic Review*, 95(4):913–935.

Kamada, Y. and Kojima, F. (2015). Efficient matching under distributional constraints: Theory and applications. *American Economic Review*, 105(1):67–99.

Kamada, Y. and Kojima, F. (2017). Stability concepts in matching under distributional constraints. *Journal of Economic Theory*, 168:107–142.

Kojima, F. (2012). School choice: Impossibilities for affirmative action. *Games and Economic Behavior*, 75(2):685–693.

Kojima, F., Tamura, A., and Yokoo, M. (2018). Designing matching mechanisms under constraints: An approach from discrete convex analysis. *Journal of Economic Theory*, 176:803–833.

Kominers, S. D. and Sönmez, T. (2016). Matching with slot-specific priorities: Theory. *Theoretical Economics*, 11(2):683–710.

Roth, A. E. (1982). The economics of matching: Stability and incentives. *Mathematics of operations research*, 7(4):617–628.

Roth, A. E. (1985). The college admissions problem is not equivalent to the marriage problem. *Journal of Economic Theory*, 36(2):277–288.

Roth, A. E. and Sotomayor, M. A. O. (1992). *Two-sided matching: A study in game-theoretic modeling and analysis*. Number 18. Cambridge university press.

Sönmez, T. (1997). Manipulation via capacities in two-sided matching markets. *Journal of Economic Theory*, 77(1):197–204.

Sönmez, T. and Ünver, M. U. (2011). Matching, allocation, and exchange of discrete resources. In *Handbook of Social Economics*, volume 1, pages 781–852. Elsevier.

Sönmez, T. and Yenmez, M. B. (2019). Affirmative action in india via vertical and horizontal reservations. *Unpublished mimeo*.

Westkamp, A. (2013). An analysis of the german university admissions system. *Economic Theory*, 53(3):561–589.