Parametric rules for state contingent claims*

Siddharth Chatterjee† Sinan Ertemel‡ Rajnish Kumar§

November 2019

Abstract

We study bankruptcy rules in a setting where individuals have state contingent claims. A rule must distribute shares before uncertainty resolves. Within a wide class of parametric rules, we first characterize rules of ex–ante form in terms of the way that the rule processes inherent uncertainty in the individual claims. The key property is: *No Penalty for Risk.* It says that the rule does not penalize an individual in a situation that differs from another only in terms of the this individual's claim in that the former situation has a risky version of the riskless claim in the latter situation. Whereas for the ex–post characterization, our key property is: *Indifference to Independent Combinations.* It says that if an individual is risk neutral with expected utility preferences then any rule that makes her indifferent between any bankruptcy problem and a corresponding independent combination of gamble between a degenerate gamble and a zero game (any bankruptcy game with zero endowment) forces the rule to be in the ex–post form. Finally, a partial comparative static result is provided which formalizes the claim that ex–ante rules are normatively more appealing for individuals when the level of the resource is low enough.

JEL code: C71, D63, D81

Key words: Rationing, Parametric rules, State contingent claims, Axiomatic characterization

1. Introduction

Consider a resource allocation problem where total claim of individuals exceeds the resource itself. How to divide a resource when individuals' objective claims cannot be honored constitutes arguably the simplest domain of distributive justice. We consider claims problem in a state contingent framework, where in the first stage every individual submits a claim corresponding each state of nature (henceforth, a "state contingent claim vector"). The realization of the state occurs in the second

*The authors express deep gratitude to Hervé Moulin and Arunava Sen for providing valuable feedbacks which has contributed in shaping the present work. Discussions with Juan Moreno–Ternero, Jens Leth Hougaard, and Ruben Juarez have been very helpful at an early stage of this work. William Thomson has kindly provided an early version of the draft of his forthcoming book on bankruptcy problems which has aided the authors to substantiate the objects investigated in this article. This work has been supported by funding from British Council grant UGC–UKIERI 2016–17–059 and the British Academy Newton Mobility grant NMG2R2–100198.

†Economics & Planning Unit, Indian Statistical Institute, 7–SJS Sansanwal Marg, New Delhi–110016, India, E–mail: 123sidch@gmail.com

‡Department of Economics, Istanbul Technical University, 34367, Maçka, Istanbul, Turkey, E–mail: ertemels@itu.edu.tr

§**Corresponding author.** Management School, Queen's University Belfast, 185 Stranmillis Road, Belfast, BT9 5EE, United Kingdom, E–mail: rajnish.kumar@qub.ac.uk

stage. There are two natural extensions of allocation rules in this setting — (1) ex–ante rules: first find expected claim of each individual and then apply the rule to the expected claim vector, and (2) ex–post rules: first find the award of each individual for each state and then take the expectation of the awards. Our main contribution is to characterize ex–ante rules within a very broad class of procedures. Additionally, we also provide a simple behavioral characterization of rules that have the ex–post form.

Ertemel & Kumar (2018) characterize ex–ante and ex–post proportional rules. Their characterization is based on the No Advantageous Reallocation (NAR) axiom, first introduced by Moulin (1985). This axiom asserts that no group of individuals can jointly benefit by reallocating their claims among themselves. By extending NAR to state contingent claims framework, along with some standard axioms in the literature, they obtain desired characterizations for proportional rules. We consider, in this paper, a more general setting where we characterize both a class of both ex–ante and all ex–post rules. To this end, we focus our attention to a natural and wide subclass of the parametric rules as introduced by Young (1987a). Parametric rules encompass most of the standard rules in the literature, such as proportional, constrained equal awards, constrained equal losses, Talmud rules and many others. Basically, parametric rules are characterized by the Consistency principle which is one of the main tenets of distributive justice. Consistency is an invariance property where some group of agents leave the problem with their awards. Accordingly, the reduced problem with the remaining individuals and remaining endowment would distribute the same awards as the original problem. We are using a weaker version of consistency which can also be interpreted as non–bossiness, an individual leaving the problem with her award, would not affect the award of the rest of the individuals.

Our second axiom is based on "Invariance to Claim Truncation" which states that a rule is invariant to truncating one individual's claim to the endowment if she claims more than the endowment itself. We further weaken this axiom by saying that an individual would not be rewarded by irrelevant claims, *i.e.*, claiming more than the endowment. Finally, we impose claim monotonicity, another standard axiom satisfied by virtually all allocation rules. It says that if an individual's claim increases then she would receive an award as much as before. By naturally extending, with substantial weakening, these axioms to state contingent claims environment, we characterize a wide class of parametric rules. Note that stochastic extensions of class of sequential priority rules, priority augmented constrained equal awards rules, and Talmud rules are members of this family.

Next, we introduce normative axioms that capture how a rule should treat all the risks inherent in the individuals' claims within the stochastic environment. The "No Penalty for Risk" axiom asserts that if an individual is faced with a riskier claim vector (*i.e.*, a mean–preserving spread of her claims) then she will *not* receive less award than before. Additionally, we impose a mild regularity condition called "No Sudden Response to Uncertainty". This axiom can be stated as follows: if a rule coincides with another rule as per which each individual's claim vector is substituted with her expected claim for some level of endowment to compute the awards, then these two rules keep being identical when the endowment level changes *slightly*. The last two axioms pin down our rule down to ex–ante form. In other words, the rule only considers the vector of expected individual claims.

Finally, we provide axiomatic characterization of ex–post rules; *i.e.*, a rule computes shares of the endowment for each individual "state–wise" and then allocates each individual an award equal to their expected share. We make no restrictions in the characterization of ex–post rules in terms of the stochastic extensions of classical rules. By taking an approach inspired by Roth (1977), which establishes Shapley Value as an Expected Utility of playing a game, we define gambles over bankruptcy games. Any bankruptcy problem can then be modelled as a combination of pairs of independent gambles — one pair corresponding to each state of nature. Each pair constitutes of degenerate gamble and a zero game (any bankruptcy game with zero endowment). An allocation rule has to be of the ex–post form, if and only if, for individuals who are expected *wealth* maximizers the rule establishes indifference between the original bankruptcy game and the corresponding independent combination of gambles.

Going back to the classical problem in the deterministic setting, the "claims problem" emerged as the intrinsic imbalance between demand and supply (*i.e.*, disequilibrium) disallowing traditional market mechanisms. Accordingly, the study of design and analysis of methods for resolving claims problems call for a normative approach constituting of an investigation of allocation rules satisfying some desiderata. There are various interpretations for claims problem such as inheritance, bankruptcy, and taxation. O'Neill (1982) provides first formal description of this problem where resource is defined as an inheritance to be distributed among heirs. Aumann and Maschler (1985), on the other hand, study bankruptcy rules where the liquidation value of a firm falls short of total claims of creditors. Young (1987a, 1987b, 1988, 1990) examines taxation schemes to determine tax liabilities for each individual according to their taxable income. Aside from these classical examples, there are obviously many more instances where a resource has to be rationed due to over demand. Emergency situations, for example, call for rationing of medical supplies and other vital good and services. A systems manager would allocate capacity, memory, and bandwidth among various nodes in the network.

In this paper, we study rationing problems where an individual's claim depends on the state of the nature where ex–post allocation is *not* feasible. Here it is essential that resource has to be divided ex–ante, namely before uncertainty is resolved. Consider a central government allocating fiscal budget among various departments and agencies. Evidently, Department of Health's budget depends on the probability of an endemic risk. Similarly, the agriculture industry is heavily dependent on the levels of actual realized rainfall. That being said, the fiscal budget allocation must be necessarily determined before the resolution of uncertainty. Going back to the "network example", the requisite bandwidth demand of the nodes depends on the stochastic network traffic. As a matter of fact uncertainty is indeed a very salient feature in claims problem. Before a systematic study of rules and normative principles in this stochastic domain, we briefly revisit some of the classic notions associated with the claims problem.

One possibility is to naively follow the "proportionality" idea; *i.e.*, each individual should receive an award *proportional* to her claim. As Aristotle pointed out in his celebrated maxim, "Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences". However, there are various axiomatic characterizations of proportional rules in claims problems such as O'Neill (1982), Moulin (1987), Chun (1988), and Ju et al. (2007) among others. Other medieval authors such as Maimonides, Ibn Ezra, and Rabad refer to Talmud and in turn reveal further normative principles in the spirit of egalitarianism. Constrained equal awards rule divides the resource equally, subject to the fact that no agent receives more than her claim. On the other hand, constrained equal losses rule equalizes the losses (*i.e.*, difference between award and claim) such that no agent receives a negative award. Axiomatic characterizations of such egalitarian rules can be found in Dagan (1996), Herrero and Villar (2001), Yeh (2008) among others. Additionally, there is a very large family of rules, sharing some characteristics from these three canonical rules. Hougaard (2009), Moulin (2002) and Thomson (2003, 2015) provide excellent surveys for allocation rules and their axiomatic characterizations.

Claims problem under uncertainty has been studied in various settings in the literature. For example, Habis and Herings (2013) extend Transferable Utility Game to stochastic environment and show that constrained equal awards rule coincides with the weak sequential core defined in Habis and Herings (2011). Xue (2018) focuses on the waste issue, namely an individual receiving an award higher than her claim. It turns out that the class of constrained equal awards rules is consistent with expected waste minimization under some normative axioms such as conditional strict endowment monotonicity, consistency, and strong upper composition. In a more recent paper, Long et al. (2019) introduces a more general division rule, *i.e.*, equal–quantile rules that similarly focuses on the waste and deficit issues. In addition to axiomatic characterization, they provide justification for this rule by maximization of a utilitarian social welfare and minimization of a utilitarian social cost function.

Our paper differs from Xue (2018) and Long et al. (2019) in modeling claim uncertainty. These papers take individual demands as a cumulative distribution function with its support being a closed interval on the real line. More importantly, they require *independence* of beliefs among the individuals. Our model, on the other hand, is more general as it allows for arbitrary joint measures as individuals' beliefs. As many real–life allocation problems exhibit correlated claims among individuals, our model provides a very robust framework to capture inherent uncertainty. In another recent paper, Hougaard and Moulin (2018) study sharing the cost in a stochastic network where the allocation should be made ex–ante before the realization of random traffic flows.

The rest of the paper is organized as follows. Section 2 describes the formal model and section 3 describes the questions tackled providing an informal indication of the nature of resolution that we offer. Section 4 describes three axioms from the bankruptcy literature as adapted naturally to the model of uncertainty. Section 5 outlines a wide class of rules to resolve bankruptcy problems in the model of uncertainty. The class defined is later shown to satisfy the axioms outlined in section 4. The key axioms, considered of various rules, that pertain to the way those rules process inherent riskiness of claims and individuals' risk posture are presented in section 6. The main results are presented in section 7. Section 8 and section 9 present the geometry of the characterization of the ex–ante rule and the strategy of the characterization of the ex–post form, respectively. The relevant logical independence of the axioms is discussed in section 10. We conclude in section 11. All proofs are relegated to the appendix which is section 12.

2. Formal model

The set of *individuals* is a nonempty finite set $N = \{1, 2, \ldots, |N|\}$ typical elements of which shall be denoted by $i, j, \ldots$ and so on. The set of *states* is a nonempty finite set S typical elements of which shall be denoted by $s_1, s_2, \ldots$ and so on. A *profile of state contingent claims matrix*, $\mathbf{c} \equiv \langle c_{is} : i \in N; s \in S\rangle$ is a map, $(i, s) \in N \times S \mapsto c_{is} \in \mathbb{R}_+$. The set of profile of state contingent claims shall be denoted by $\mathscr{C}$. For any individual i, $\mathbf{c}_i \equiv \langle c_{is} : s \in S\rangle$ is *individual i's state contingent claim vector*; *i.e.*, the map $s \in S \mapsto c_{is} \in \mathbb{R}_+$ as obtained by restriction of the map $\mathbf{c}$ to the set $\{i\} \times S$. For any state $s \in S$, $\mathbf{c}_s \equiv \langle c_{is} : i \in N\rangle$ is the *profile of claims in state s*; *i.e.*, the map $i \in N \mapsto c_{is} \in \mathbb{R}_+$ as obtained by restriction of the map $\mathbf{c}$ to the set $N \times \{s\}$. An *estate* is any element of $\mathscr{E} := \mathbb{R}_+$ typically denoted by $E, E', \ldots$ or $E_1, E_2, \ldots$ and so on. An *assessment of state probabilities*, denoted by $\mathbf{p} \equiv \langle p_s : s \in S \rangle$, is a map, $s \in S \mapsto p_s \in [0,1]$ such that $\sum_{s \in S} p_s = 1$. Thus, the set of assessments of state probabilities is the $|S| - 1$ dimensional simplex $\Delta(S)$; *i.e.*, $\mathbf{p} \in \Delta(S)$. For any $s \in S$, $\boldsymbol{\delta}_s \in \Delta(S)$ shall denote the lottery which is degenerate at the state s. For the map $\mathbf{c}_i$ and $\mathbf{p} \in \Delta(S)$, define $\bar{c}_i(\mathbf{p}) \coloneqq \sum_{s \in S}(p_s \cdot c_{is})$; *i.e.*, the *expected claim of individual* i. We shall follow the convention that, for any set K, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^K$, $[\mathbf{x} \geq \mathbf{y}] \iff (\forall k \in K)[x(k) \geq y(k)]$.[1] Also, $\mathbf{0}_K$ shall denote the map, $k \in K \mapsto 0 \in \mathbb{R}_+$ and $\mathbf{1}_K$ shall denote the map, $k \in K \mapsto 1 \in \mathbb{R}_+$.[2]

A *bankruptcy problem* is an ordered triple $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{C} \times \mathscr{E} \times \Delta(S)$ such that, for every $s \in S$, $\sum_{i \in N} c_{is} \geq E$. The set of bankruptcy problems, $\mathscr{D} \coloneqq \{(\mathbf{c}, E, \mathbf{p}) \in \mathscr{C} \times \mathscr{E} \times \Delta(S) : (\forall s \in S)[\sum_{i \in N} c_{is} \geq E]\}$ shall be called the *domain*. A *rule* is a map, $\phi : \mathscr{D} \to \mathbb{R}^N_+$, continuous in resource, such that, for any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, $\sum_{i \in N}[\phi(\mathbf{c}, E, \mathbf{p})](i) = E$.[3] The terms "$[\phi(\mathbf{c}, E, \mathbf{p})](i)$" and "$\phi_i(\mathbf{c}, E, \mathbf{p})$" shall be used interchangeably. $\phi_i(\mathbf{c}, E, \mathbf{p})$ shall be called the *share* of individual i in the bankruptcy problem $(\mathbf{c}, E, \mathbf{p})$ according to the rule ϕ. Denote by $\mathscr{D}^*$ the set $\{(\mathbf{x}, t) \in \mathbb{R}^N_+ \times \mathbb{R}_+ : \sum_{i \in N} x_i \geq t\}$. We define any rule ϕ to be *ex–ante*, if and only if, there exists a corresponding function $\psi : \mathscr{D}^* \to \mathbb{R}^N_+$ such that $\sum_{i \in N} \psi_i(\mathbf{x}, t) = t$ for any $(\mathbf{x}, t) \in \mathscr{D}^*$, and $\phi(\mathbf{c}, E, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E)$ for any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ where $\bar{c}(\mathbf{p}) \coloneqq \langle \bar{c}_i(\mathbf{p}) : i \in N \rangle$. A rule ϕ is defined to be *ex–post*, if and only if, for any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, $\phi(\mathbf{c}, E, \mathbf{p}) = \sum_{s \in S} p_s \cdot \phi(\mathbf{c}, E, \boldsymbol{\delta}_s)$.

Denote by $\mathscr{V}$ the set $\mathscr{C} \times \mathscr{E}$. An element of $\mathscr{V}$, $\mathbf{v} \equiv (\mathbf{c}, E)$, shall be called a *situation*. For any situation $\mathbf{v} \equiv (\mathbf{c}, E)$, the corresponding situation $\mathbf{v}_0 \coloneqq (\mathbf{c}, 0)$ shall be called the *corresponding zero situation*. For $\mathbf{v} \equiv (\mathbf{c}, E)$ and $\mathbf{p} \in \Delta(S)$, we shall identify $(\mathbf{v}, \mathbf{p})$ with $(\mathbf{c}, E, \mathbf{p})$. Let $\mathscr{D}$ be endowed with the σ–algebra, $\mathscr{F}_{\mathscr{D}}$, generated by the class of all finite subsets[4] of $\mathscr{D}$. For any $K \in \mathbb{N}$, let $\pi_1, \pi_2, \ldots, \pi_K \in [0,1]$ such that $\sum_{k=1}^K \pi_k = 1$, and consider any K bankruptcy problems $(\mathbf{v}_1, \mathbf{p}_1), (\mathbf{v}_2, \mathbf{p}_2), \ldots, (\mathbf{v}_K, \mathbf{p}_K) \in \mathscr{D}$. Then $[\bigoplus_{k=1}^K \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)]$ and $[\pi_1 \bullet (\mathbf{v}_1, \mathbf{p}_1) \bigoplus \pi_2 \bullet (\mathbf{v}_2, \mathbf{p}_2) \bigoplus \ldots \bigoplus \pi_K \bullet (\mathbf{v}_K, \mathbf{p}_K)]$ shall denote the lottery with outcomes $(\mathbf{v}_1, \mathbf{p}_1), (\mathbf{v}_2, \mathbf{p}_2), \ldots, (\mathbf{v}_K, \mathbf{p}_K)$ in $\mathscr{D}$ having probabilities $\pi_1, \pi_2, \ldots, \pi_K$, respectively. Formally, we have the probability measure $[\bigoplus_{k=1}^K \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)]$, over the measurable space $(\mathscr{D}, \mathscr{F}_{\mathscr{D}})$, which is the map $D \in \mathscr{F}_{\mathscr{D}} \mapsto [\bigoplus_{k=1}^K \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)](D) \coloneqq \sum_{k \in \{1,2,\ldots,K\} : (\mathbf{v}_k, \mathbf{p}_k) \in D} \pi_k$. Let $\Delta(\mathscr{D})$ be the family of all probability measures, over $(\mathscr{D}, \mathscr{F}_{\mathscr{D}})$, with finite supports. Thus, elements of $\Delta(\mathscr{D})$ are precisely objects of the form

[1] We may interchangeably write "x_k" for "$x(k)$", and "y_k" for "$y(k)$". Thus, all we are saying is $[\mathbf{x} \geq \mathbf{y}] \iff (\forall k \in K)[x_k \geq y_k]$. This is the usual partial order over $\mathbb{R}^K$ of "a vector dominating another componentwise" except that the set K is abstract.

[2] That is, $\mathbf{0}_K$ is the "vector of zeroes in $\mathbb{R}^K_+$", and $\mathbf{1}_K$ is the "vector of ones in $\mathbb{R}^K_+$".

[3] Note that $\mathbb{R}^N_+$ is the set of all maps with domain N and codomain $\mathbb{R}_+$. Thus, for $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, $\phi(\mathbf{c}, E, \mathbf{p})$ is indeed a map from N to $\mathbb{R}_+$. Since $i \in N$, one obtains $[\phi(\mathbf{c}, E, \mathbf{p})](i) \in \mathbb{R}_+$. Also, see footnote 4.

[4] That is, the countable–cocountable σ–algebra. By definition, this is the class of subsets that are either countable or have countable complement.

$[\bigoplus_{k=1}^{K} \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)]$. Each element of $\Delta(\mathscr{D})$ shall be called a *gamble*. The gamble $[\bigoplus_{k=1}^{K} \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)]$, under the rule ϕ, induces the *money lottery* $[\bigoplus_{k=1}^{K} \pi_k \bullet \phi_i(\mathbf{v}_k, \mathbf{p}_k)]$ for each individual i. That is, the gamble in which, for each $k \in \{1, 2, \ldots, K\}$, the bankruptcy problem $(\mathbf{v}_k, \mathbf{p}_k)$ is played with probability π_k, provides the individual i the share $\phi_i(\mathbf{v}_k, \mathbf{p}_k)$ with probability π_k.

For any $M \in \mathbb{N}$, let $\mu_1, \mu_2, \ldots, \mu_M \in \Delta(\mathscr{D})$. Then $\mu_1 \bigotimes \mu_2 \bigotimes \ldots \bigotimes \mu_M$ and $\bigotimes_{m=1}^{M} \mu_m$ are the names of the product measure, over the product space $(\bigtimes_{m=1}^{M} \mathscr{D}, \bigotimes_{m=1}^{M} \mathscr{F}_{\mathscr{D}})$, obtained from the underlying probability spaces. Any such measure $\bigotimes_{m=1}^{M} \mu_m$ will be called an *independent combination of gambles*. Let, for each $m \in \{1, 2, \ldots, M\}$, μ_m be the gamble $[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]$. Then the independent combination of gambles $\bigotimes_{m=1}^{M} \mu_m$ provides each individual i the share $\sum_{m=1}^{M} \phi_i(\mathbf{v}_{k_m}^m, \mathbf{p}_{k_m}^m)$ with probability $\prod_{m=1}^{M} \pi_{k_m}^m$. The class of all independent combinations of gambles shall be denoted by $\mathscr{I}_{\mathscr{D}}$. For each individual i, driven by von Neumann–Morgenstern preferences over money lotteries, let $\succsim_i$ be the complete and transitive binary relation over $\mathscr{I}_{\mathscr{D}}$. Formally, $\succsim_i$ is defined, over $\mathscr{I}_{\mathscr{D}}$, as follows. Let $u_i : \mathbb{R}_+ \to \mathbb{R}$ be any Bernoullian whose expected utility represents individual i's preferences. Define a map $U : \mathscr{I}_{\mathscr{D}} \to \mathbb{R}$ as follows. Let $\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]$ be an element of $\mathscr{I}_{\mathscr{D}}$. Then define:

$$U\left(\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]\right) := \sum_{\langle k_m \le K_m : m \le M\rangle} [\prod_{m=1}^{M} \pi_{k_m}^m] \cdot u_i\Big(\sum_{m=1}^{M} \phi_i(\mathbf{v}_{k_m}^m, \mathbf{p}_{k_m}^m)\Big).$$

For any two independent combinations of gambles $\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]$ and $\bigotimes_{m=1}^{M'}[\bigoplus_{k=1}^{K'_m} {\pi'}_k^m \bullet ({\mathbf{v}'}_k^m, {\mathbf{p}'}_k^m)]$, we define $\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)] \succsim_i \bigotimes_{m=1}^{M'}[\bigoplus_{k=1}^{K'_m} {\pi'}_k^m \bullet ({\mathbf{v}'}_k^m, {\mathbf{p}'}_k^m)]$, if and only if, the following holds:

$$U\left(\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]\right) \ge U\left(\bigotimes_{m=1}^{M'}[\bigoplus_{k=1}^{{K'}_m} {\pi'}_k^m \bullet ({\mathbf{v}'}_k^m, {\mathbf{p}'}_k^m)]\right).$$

This completes the specification of the formal model and the notations.

3. Core problems

First, we shall offer characterizations for two wide classes of rules, defined in the context of uncertainty as above, that are insensitive to the risk that may be inherent to the state contingent claim which some of the individuals may face in a bankruptcy problem. A rule is of the *ex–ante* form if it takes into consideration only the vector of expected individual claims for any bankruptcy problem. Our first theorem says that within the class of rules that satisfy claim monotonicity together with some weakenings of consistency and truncation of irrelevant claims, those which do not penalize an individual for facing risk, under a mild regularity condition, are precisely the ones that are ex–ante. The second theorem asserts that the same property of the rule towards risk inherent in claims characterizes the ex–ante form within the class of rules that satisfy axioms which are dual to those which define the wider class of the first theorem.

Second, we characterize rules, which initially compute shares "state–wise" and then award every individual the resulting expectation. Such rules are said to be of the *ex–post* form for they process each bankruptcy problem as the expectation of the corresponding collection of "ex–post problems". The basis of this characterization is how the rule ϕ preserves an individual's preference over different independent combinations of gambles. Consider, individual i faces the bankruptcy problem $(\mathbf{c}, E, \mathbf{p})$. Alternately, consider that the individual faces an independent combination of gambles in which, *independently*, with probability p_s she faces the problem $(\mathbf{c}, E, \boldsymbol{\delta_s})$ and with probability $1 - p_s$ she faces the problem $(\mathbf{c}, 0, \boldsymbol{\delta_s})$. The individual observes that in the first problem, the realization of any state excludes the realization of the other states; *i.e.*, if the "ex–post problem" $(\mathbf{c}, E, \boldsymbol{\delta_s})$ is realized, then $(\mathbf{c}, E, \boldsymbol{\delta_{s'}})$ *cannot*. However, such is *not* the case in the second problem due to *independence* in the combination. Also, the probabilities induced over each of the "ex–post problems", $(\mathbf{c}, E, \boldsymbol{\delta_s})$ is the same in both. Further, the rule ϕ induces a money lottery, for the individual, corresponding to each of the problems. The individual, with von Neumann–Morgenstern preferences over money lotteries, being risk neutral ranks all independent combinations of gambles using the expected return by the rule ϕ. Our characterization is that the rule has the ex–post form precisely when it enforces the individual to be indifferent between any bankruptcy problem and the corresponding independent combination of gambles.

4. Some standard axioms

Three prominent axioms shall be considered, from the bankruptcy literature, and adapted to the model of uncertainty. The statement of each of the three axioms is followed by a brief discussion of the strength and normative appeal. In addition to Hougaard (2009), Moulin (2002) and Thomson (2003, 2015), for a comprehensive account, we draw the reader's attention to the forthcoming monograph by Thomson (2019).

Claim Monotonicity. If $(\mathbf{c}, E, \mathbf{p}), ((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) \in \mathscr{D}$ and $i \in N$ such that $\mathbf{c}_i \leq \mathbf{c}'_i$, then $\phi_i(\mathbf{c}, E, \mathbf{p}) \leq \phi_i((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$.

The above axiom is the most obvious adaptation of the "claim monotonicity" axiom from the bankruptcy literature with deterministic claims. In words, consider two profiles of state contingent claims such that the profiles differ only in one specific individual's state contingent claim and that too in the sense that the claims, of the individual, in the second profile are greater than that in the first state–wise. The rule satisfies the axiom, if and only if, it provides that individual at least as much in the second profile as it does in the first. The "claim monotonicity" property, in the deterministic setting, holds for many major rules such as a Priority Augmented Weighted Constrained Equal Awards rules, the Talmud rule, the Proportional rule and so on. However, formally there exists several rules which do not satisfy this property as can be seen from the definitions of two particularly wide class of rules which are the class of *fixed path rules* and Young's class of *parametric rules*. The first class is important in the characterization of the rules that satisfy the property of "independence of irrelevant claims" while the second class characterizes the property of "consistency". In particular, both the classes admit as nonempty proper subclasses of rules that either do satisfy "claim monotonicity" or do not. The adapted form for the setting with uncertainty has the same normative justification as is for its deterministic version — it is natural to expect that a proposed rule, fixing the claims of every other individual, does not give an individual less when in fact her claim is higher. Notice, that this normative justification is adequate in the axiomatic framework which does not consider individuals to be strategic; that is, all individuals are assumed to be truthful in terms of reporting their respective claims so that the profile of individual claims is common knowledge among all the individuals and the planner.

Weak Consistency. If $(\mathbf{c}, E, \mathbf{p}), ((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) \in \mathscr{D}$ and $i \in N$, then $\sum_{j \in N \setminus \{i\}} \phi_j(\mathbf{c}, E, \mathbf{p}) = \sum_{j \in N \setminus \{i\}} \phi_j((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$ implies $\phi_j(\mathbf{c}, E, \mathbf{p}) = \phi_j((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$ for every $j \in N \setminus \{i\}$.

The property called "consistency" is of wide appeal in the bankruptcy literature with deterministic claims. Many of the major rules satisfy the property of "consistency". For instance, the class of Priority Augmented Weighted Constrained Equal Awards rules, the Proportional rule and the Talmud rule. In fact, every rule in Young's class of *parametric rules* satisfies "consistency". However, the Random Order of Arrival rule does not satisfy this property. To briefly recall the idea of "consistency", consider the shares computed by the rule for a problem involving some set of individuals. Next, a group of some of the individuals leaves having obtained their shares. The property demands that the rule allocates the same shares from the sum of the shares of the remaining individuals as it had computed initially. The above axiom is not a direct adaptation of the "consistency" property to the setting involving uncertainty. In fact, all it requires is that given that the sum of individual shares leaving one in two profiles, as computed by the rule, is the same, if the two profiles differ in terms of the state contingent claim of only *that* individual, then the shares of the other individuals is the same in both the profiles. Since the above axiom is strictly weaker than "consistency", we have named this property as such. We admit here that the role of the above axiom in our characterization of the ex–ante form is only substantial when there are at least three individuals. This is because the above axiom holds vacuously in a setting with only two individuals due to "budget balancedness"; that is, the fact that the rule must *fully* divide the resource into individual shares.

No Reward for More Irrelevant Claims. If $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ and $i \in N$ such that $\min_{s \in S} c_{is} \geq E$, then $\delta \mathbf{c}_i \geq \mathbf{0}_S$ implies $\phi_i(\mathbf{c}, E, \mathbf{p}) \geq \phi_i((\mathbf{c}_i + \delta \mathbf{c}_i, \mathbf{c}_{-i}), E, \mathbf{p})$.

The above axiom is an adaptation to the setting with uncertainty of a slight weakening of the "independence of irrelevant claims" property which is also known as "truncation of irrelevant claims". The idea is, given individual claims and a resource, any claim matters only as long as it does not exceed the resource. If an individual's claim does exceed the resource, then her claim is "truncated" in the sense that the rule considers her claim to be the level of the resource itself. Many important rules satisfy "truncation of irrelevant claims". For instance, the class of Priority Augmented Weighted Constrained Equal Awards rules, the Talmud rule, the Minimal Overlap rule and so on. In fact, the class of *fixed path rules* admits a large subclass each element of which satisfies the property. However, some important rules do not satisfy "truncation of irrelevant claims". Examples include the Proportional Rule and the Constrained Equal Losses rule. We claim that our axiom is an adaptation of a mild weakening of the "truncation of irrelevant claims" property. To see this note, the above property only demands that if every of the state-wise claims of an individual exceed the resource, then the individual does not gain more than what she would had her state–wise claims equalled the resource. We note that the combination of *claim monotonicity* and *no reward for more irrelevant claims* implies the natural adaptation of the "truncation of irrelevant claims" property to the setting with uncertain claims.

5. A class of parametric rules

The purpose of this section is to define a class of rules over the domain $\mathscr{D}$, which shall be denoted by

Φ, such that any rule $\phi \in \Phi$ satisfies each of the standard axioms described in section 4. Our claim is that the class Φ is a *subclass* of all those rules which satisfy the axioms. That shall be the content of Theorem 1 stated in the section 6. The definition of the class Φ requires some preliminaries. The abstract definition of the class shall be discussed to make a case that it is a very large class. Members of the class Φ are constructed by the composition of rules, from a family contained in the class of Young's "parametric rules", with a profile of $\mathbb{R}$–valued functions, which map pairs $(\mathbf{c}, \mathbf{p}) \in \mathscr{C} \times \Delta(S)$ satisfying some conditions. Typical profiles shall be denoted by $T \equiv \langle T_i : i \in N \rangle$, and the class of all such profiles shall be denoted by $\mathscr{T}$.

Let $T \equiv \langle T_i : i \in N \rangle \in \mathscr{T}$, if and only if, for every $i \in N$, $T_i : \mathscr{C} \times \Delta(S) \to \mathbb{R}$ is a map, and, for any $i \in N$ and any $(\mathbf{c}, \mathbf{p}) \in \mathscr{C} \times \Delta(S)$, each of the following conditions hold:

$R.1$ $\sum_{i \in N} T_i(\mathbf{c}, \mathbf{p}) \geq \min_{s \in S} \sum_{i \in N} c_{is}$.

$R.2$ $T_i(\mathbf{c}, \mathbf{p}) \geq \min_{s \in S} c_{is}$.

$R.3$ If $\mathbf{c}'_i > \mathbf{c}_i$, then $T_i((\mathbf{c}'_i, \mathbf{c}_{-i}), \mathbf{p}) > T_i(\mathbf{c}, \mathbf{p})$.

Recall that $\mathscr{D}^* = \{(\mathbf{x}, t) \in \mathbb{R}^N_+ \times \mathbb{R}_+ : \sum_{i \in N} x_i \geq t\}$ is the class of determinsitic bankruptcy problems. As indicated prior to the definition of $\mathscr{T}$, since profiles of form $T \in \mathscr{T}$ shall be deployed as a proxy for "claims" in deterministic rules, we enforce condition $R.1$ to ensure that $(T(\mathbf{c}), E) \in \mathscr{D}^*$ whenever $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. Looking ahead, the condition $R.2$ is enforced to assist the achievement of the *no reward for more irrelevant claims*. Observe that $R.2$ ensures: if $i \in N$ and $\min_{s \in S} c_{is} \geq E$, then $T_i(\mathbf{c}, \mathbf{p}) \geq E$. Thus, if a deterministic rule acts on the profile T, then the resulting rule satisfies *no reward for more irrelevant claims* whenever the deterministic rule satisfies "independence of claims truncation". Finally, condition $R.3$ is enforced to assist in obtaining *claims monotonicity.* In particular, if the determinisitic rule chosen satisfies "claim monotonicity", then the rule produced by its composition with T satisfies *claim monotonicity* if $R.3$ holds. As will be seen, this condition also assists in ensuring that the resulting rule satisfies *weak consistency.*

Observe, the class $\mathscr{T}$ is closed under (finite) convex combinations. The class $\mathscr{T}$ is indeed very large. To create a family of concrete examples of $T \in \mathscr{T}$, for any $r \geq 1$, let $\|(\mathbf{x}, \mathbf{p})\|_r \coloneqq (\sum_{s \in S} p_s \cdot x_s^r)^{1/r}$ where $\mathbf{x} \equiv \langle x_s \in \mathbb{R}_+ : s \in S \rangle$. Now, fix any $(\mathbf{c}, \mathbf{p}) \in \mathscr{C} \times \Delta(S)$. For every $i \in N$, let $K_i \in \mathbb{N}$. For any fixed $1 \leq r_1 < r_2 < \ldots < r_{K_i}$ and $\alpha_0^i(\mathbf{p}), \alpha_1^i(\mathbf{p}), \ldots, \alpha_{K_i}^i(\mathbf{p}) \in [0, 1]$ such that $\sum_{k=0}^{K_i} \alpha_k^i(\mathbf{p}) = 1$, define:

$$T_i(\mathbf{c}, \mathbf{p}) \coloneqq \alpha_0^i(\mathbf{p}) \cdot \max_{s \in S} c_{is} + \sum_{k=1}^{K_i} \alpha_k^i(\mathbf{p}) \cdot \|(\mathbf{c}_i, \mathbf{p})\|_{r_k}$$

Thus, we have defined a profile $T \coloneqq \langle T_i : i \in N \rangle$ of maps, where $T_i : \mathscr{C} \times \Delta(S) \to \mathbb{R}$ for every $i \in N$. Now, for any $\mathbf{x} \in \mathbb{R}^S$, we have $\max_{s \in S} x_s \geq x_s \geq \min_{s \in S} x_s$ for any $s \in S$. We obtain:

$S.1$ $\sum_{i \in N} \max_{s \in S} c_{is} \geq \sum_{i \in N} c_{is} \geq \min_{s \in S} \sum_{i \in N} c_{is}$ for any $s \in S$.

$S.2$ $\max_{s \in S} c_{is} \geq \min_{s \in S} c_{is}$.

$S.3$ If $\mathbf{c}'_i > \mathbf{c}_i$, then $\max_{s \in S} c'_{is} > \max_{s \in S} c_{is}$.

Also, for any $r \geq 1$, by Jensen's inequality, we have $\|(\mathbf{x}, \mathbf{p})\|_r \geq \|(\mathbf{x}, \mathbf{p})\|_1$, as the map $\zeta \in \mathbb{R}_+ \mapsto \zeta^{1/r} \in \mathbb{R}_+$ is convex. Now, $\|(\mathbf{x}, \mathbf{p})\|_1 \geq \min_{s \in S} x_s$. Thus, we obtain: $\|(\mathbf{c}_i, \mathbf{p})\|_r \geq \min_{s \in S} c_{is}$. Further, $\sum_{i \in N} \|(\mathbf{c}_i, \mathbf{p})\|_1 = \sum_{s \in S} (p_s \cdot [\sum_{i \in N} c_{is}]) \geq \min_{s \in S} \sum_{i \in N} c_{is}$. Hence, $\sum_{i \in N} \|(\mathbf{c}_i, \mathbf{p})\|_r \geq \min_{s \in S} \sum_{i \in N} c_{is}$. Hence, for any $k \in \{1, 2, \ldots, K_i\}$, we have:

$T.1$ $\sum_{i \in N} \|(\mathbf{c}_i, \mathbf{p})\|_{r_k} \geq \min_{s \in S} \sum_{i \in N} c_{is}$.

$T.2$ $\|(\mathbf{c}_i, \mathbf{p})\|_{r_k} \geq \min_{s \in S} c_{is}$.

$T.3$ If $\mathbf{c}'_i > \mathbf{c}_i$, then $\|(\mathbf{c}'_i, \mathbf{p})\|_r > \|(\mathbf{c}_i, \mathbf{p})\|_r$.

where the last one is obvious. Observe, for any fixed $(\mathbf{c}, \mathbf{p})$, any convex combination of the corresponding inequalities, from the lists $\langle S.1, S.2, S.3 \rangle$ and $\langle T.1, T.2, T.3 \rangle$, holds. Thus, $T \in \mathscr{T}$. Recall, we observed that the class $\mathscr{T}$ is closed under convex combinations. Given the choices available in the definition of the profile T, it is clear that indeed $\mathscr{T}$ is very large.

Now, we shall specify a certain subclass of rules inspired by Young's "parametric rules". Let $h \equiv \langle h_i : i \in N \rangle \in \mathscr{H}$, if and only if, there exists $\theta_*, \theta^* \in \mathbb{R}$ with $\theta_* < \theta^*$ such that, for every $i \in N$, $h_i : [\theta_*, \theta^*] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies each of the following:

$H.1$ For any $x \in \mathbb{R}_+$, $h_i(\theta_*, x) = 0$ and $h_i(\theta^*, x) = x$.

$H.2$ For any $x \in \mathbb{R}_+$, the map $\theta \in [\theta_*, \theta^*] \mapsto h_i(\theta, x)$ is continuous.

$H.3$ For any $x \in \mathbb{R}_+$, the map $\theta \in [\theta_*, \theta^*] \mapsto h_i(\theta, x)$ is strictly increasing.

$H.4$ For any $x, x' \in \mathbb{R}_+$ with $x < x'$ and any $\theta \in [\theta_*, \theta^*]$, $h_i(\theta, x) < h_i(\theta, x')$.

For any $h \in \mathscr{H}$, we define a corresponding map $\psi^h : \mathscr{D}^* \to \mathbb{R}^N_+$ as follows. For every $i \in N$, and for any $(\mathbf{x}, t) \in \mathscr{D}^*$, let $\psi^h_i(\mathbf{x}, t) \coloneqq h_i(\theta, \min\{x_i, t\})$ where $\theta \in [\theta_*, \theta^*]$ solves $\sum_{i \in N} h_i(\theta, \min\{x_i, t\}) = t$. Set $\psi^h \coloneqq \langle \psi^h_i : i \in N \rangle$. Observe, for any problem $(\mathbf{x}, t) \in \mathscr{D}^*$, the resulting profile of "truncated claims" $\mathbf{x}^t \coloneqq \langle \min\{x_i, t\} : i \in N \rangle$ defines some $\theta \in [\theta_*, \theta^*]$ that solves $\sum_{i \in N} h_i(\theta, x^t_i) = t$ by the properties $H.1$ and $H.2$. That such a solution is unique follows from property $F.3$. That is, ψ^h is indeed a rule over the domain $\mathscr{D}^*$ if $h \in \mathscr{H}$. That only the "truncated claims" are processed by ψ^h is obvious. Hence, ψ^h satisfies "independence of irrelevant claims". Property $H.4$ is enforced to ensure that ψ^h satisfies "claim monotonicity". This point is presented in detail in the proof of Theorem 1 in the appendix. Finally, observe that ψ^h satisfies "consistency". This case follows from the observations that, given a level of the resource t, the rule ψ^h only processes "truncated claims" and with the fixed resource level the rule is a Young's "parametric rule". Recall, the specialization of *weak consistency* to the setting of detereministic problems is strictly weaker than "consistency".

Given the above preliminaries, observe that, for any $T \in \mathscr{T}$ and any $h \in \mathscr{H}$, the corresponding map $\phi^{h,T} : \mathscr{D} \to \mathbb{R}^N_+$ defined by:

$$\phi^{h,T}(\mathbf{c}, E, \mathbf{p}) \coloneqq \psi^h(T(\mathbf{c}, \mathbf{p}), E), \text{ for every } (\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}.$$

Based on the above discussion, it follows that $\phi^{h,T}$ is a rule, in the setting of bankruptcy problems with uncertainty, such that the properties *weak consistency*, *claim monotonicity* and *no reward for more irrelevant claims* holds. We are now in a position to define the class of rules Φ. Let $M \in \mathbb{N}$. For every $\mathbf{p} \in \Delta(S)$, let $\beta_1(\mathbf{p}), \beta_2(\mathbf{p}), \ldots, \beta_M(\mathbf{p}) \in [0,1]$ such that $\sum_{m=1}^{M} \beta_m(\mathbf{p}) = 1$. Also, for each $m \in \{1, 2, \ldots, M\}$, let $h_m \in \mathscr{H}$ and $T_m \in \mathscr{T}$. Define $\phi : \mathscr{D} \to \mathbb{R}^N_+$ as follows:

$$\phi(\mathbf{c}, E, \mathbf{p}) \coloneqq \sum_{m=1}^{M} \beta_m(\mathbf{p}) \cdot \phi^{h_m, T_m}(\mathbf{c}, E, \mathbf{p}), \text{ for every } (\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}.$$

It is immediate that ϕ so defined is indeed a rule satisfying all the three axioms. Then Φ is defined to be the collection of rules precisely of the form of ϕ. Theorem 1, stated in section 7, formally records this conlcusion of the present section.

Before closing this section, we point out two particular sublasses of Φ. The first one is a subclass of rules that have the ex–ante form. To see that this is indeed the case, we consider $T \in \mathscr{T}$ as follows. Let $T \equiv \langle T_i : i \in N \rangle$ where, for every individual $i \in N$, the map $T_i : \mathscr{C} \times \Delta(S) \to \mathbb{R}_+$ is defined as: $T_i(\mathbf{c}, \mathbf{p}) \coloneqq \|(\mathbf{c}_i, \mathbf{p})\|_1$. Recall, $\|(\mathbf{c}_i, \mathbf{p})\|_1 = \sum_{s \in S} p_s \cdot c_{is}$ by definition. Thus, for any choice of $h \in \mathscr{H}$, the resulting rule $\phi^{h,T}$ has the ex–ante form. We shall call this rule the *ex–ante rule defined by h*. Given the definition of the class $\mathscr{H}$, the ex–ante versions of many prominent rules such as Talmud rule, Minimal Overlap rule and the family of Priority Augmented Weighted Constrained Equal Awards rules studied in Flores-Szwagrzak (2015) which includes Constrained Equal Awards rule and weighted Constrained Equal Awards rules are contained in the class Φ. Some rules are, however, not in Φ. For instance, Proportional rule, Constrained Equal Losses rule, Reverse Talmud rules (van den Brink et al. 2013, van den Brink and Moreno-Ternero 2017) and

Random Order of Arrival rule do not belong to this class.

Next, we observe that a subclass of rules, having the ex–post form, are also contained in Φ. Define $M := |S|$. For every $s \in S$, we define $T_s \in \mathscr{T}$ as follows. Fix $s \in S$. Let $T_s \equiv \langle T_{s,i} : i \in N \rangle$, where, for any $i \in N$, $T_{s,i}(\mathbf{c}, \mathbf{p}) := c_{is}$ for every $(\mathbf{c}, \mathbf{p}) \in \mathscr{C} \times \Delta(S)$. Also, for any $s \in S$, define $\beta_s(\mathbf{p}) := p_s$ for every $\mathbf{p} \in \Delta(S)$. Clearly, $\sum_{s \in S} \beta_s(\mathbf{p}) = 1$. Fix any $h \in \mathscr{H}$, and define the map $\phi : \mathscr{D} \to \mathbb{R}^N_+$ as follows:

$$\phi(\mathbf{c}, E, \mathbf{p}) := \sum_{s \in S} \beta_s(\mathbf{p}) \cdot \psi^{h, T_s}(T(\mathbf{c}, \mathbf{p}), E), \text{ for every } (\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}.$$

It follows that ϕ, so defined, is in the class Φ. Notice, by construction, rule ϕ has the ex–post form. We shall call this rule the *ex–post rule defined by h*. In particular, it follows that the ex–post versions of Talmud rule, Minimal Overlap rule and any member of the Priority Augmented Weighted Constrained Equal Awards rules are elements of the class Φ. Again, the Constrained Equal Losses rule and the Random Order of Arrival rule do not belong to this class.

6. Axioms on rules pertaining to risky claims and individuals' risk posture

First, we present the axioms that are relevant in the characterization theorems of rules that have the ex–ante form. The formal statement of each axiom is followed with a brief discussion regarding its interpretation, normative justification and remarks on the strength. The two axioms describe aspects, a rule may satisfsy, regarding the way uncertainty inherent in claims may be treated.

No Penalty for Risk. If $(\mathbf{c}, E, \mathbf{p}), ((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) \in \mathscr{D}$ and $i \in N$ such that $\mathbf{c}'_i = \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S$, then $\phi_i(\mathbf{c}, E, \mathbf{p}) \geq \phi_i((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$.

To see the interpretation of the above axiom, fix the claims of every other individual and consider two state contingent claims of the individual that differ only in that the first claim is "risky" while the second is not. In particular, the first claim is a mean–preserving spread of the second. Observe, the notion of "riskier claim" is equivalent to second–order stochastic dominance which in turn is much weaker than the notion of mean–preserving spread. This is so as second–order stochastic dominance is obtained by any *sequence* of mean–preserving spreads. As a result, the axiom has in its hypothesis a predicate that is strictly stronger than what it should be as per the suggestion of its name. That is, the axiom is indeed weaker than what its name seems to suggest. Then the rule that satisfies the above property does not allocate the individual less in the first profile than what it would in the second profile. The normative justification for this stems from the idea that if the planner proposes a rule to individuals, who may later potentially find themselves in a bankruptcy problem, then the individuals would care to have a rule that does not punish them just because they face a risky version of some claim. In fact, as the upside of risk being large rewards often is the motivation for individuals to invest or participate in situations with a potential for limited resource, a assurance of no punishment for undertaking risk is perhaps more welcome on part of the individuals to whom the rule is being proposed by the planner. Note, the axiom does not demand that the two state contingent claims of the individual be treated identically. That such would be indeed the case — the rule has the *ex–ante* form — shall be seen to follow in large part from this axiom. See the statement of Lemma 1, for more on this. That any rule with the ex–ante form indeed satisfies this axiom is trivial.

No Sudden Response to Uncertainty. If $\mathbf{c}, \mathbf{c}' \in \mathscr{C}$ and $E^\dagger \in \mathbb{R}_{++}$ such that $\mathbf{c}'_i = \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S$ for every $i \in N$, and $(\mathbf{c}, E^\dagger, \mathbf{p}), (\mathbf{c}', E^\dagger, \mathbf{p}) \in \mathscr{D}$, then $\phi(\mathbf{c}, E, \mathbf{p}) = \phi(\mathbf{c}', E, \mathbf{p})$ for every $E \leq E^\dagger$ implies that $\phi(\mathbf{c}, E, \mathbf{p}) = \phi(\mathbf{c}', E, \mathbf{p})$ for any E in some neighborhood of $E^\dagger$.

For interpreting the above axiom, we consider two state contingent profiles of individual claims as follows. The second of the two is the "equivalent deterministic profile" in that every individual's claim is the same across every state of nature equalling the risk free mean claim. Now, suppose it is case that the two profiles are treated identically by the rule in that every individual gets the same in both the profiles under the rule as long as the resource is up to some strictly positive level of the resource. Then the rule satisfies the above axiom, if and only if, the rule continues to process the two profiles identically in some neighborhood of that level. The essence that the axiom captures is that the rule does not start to show a response *suddenly* to the inherent uncertainty that may be present in individual claims of the first profile. This achieves some regularity in the response of the rule to "indeterminacy". Such a property is of normative appeal to the individuals, who know that they may potential have claims with inherent uncertainty, to whom the planner has to propose a rule to divide. Observe, for this axiom to have any bite, it is essential to establish, given a state contingent profile of individual claims and the "equivalent deterministic profile", the rule process the two profiles identically for every possible resource up to a non–trivial level.

Now, we present the only axiom that characterizes the ex–post form of rules. Again, the formal statement of the axiom is followed with a brief discussion regarding its interpretation, normative justification and remarks on the strength.

Indifference to Independent Combinations. If $(\mathbf{v}, \mathbf{p}) \in \mathscr{D}$ and $\mathbf{q} \in \Delta(S)$, then $\bigotimes_{s\in S}[p_s \bullet (\mathbf{v}, \boldsymbol{\delta}_s) \bigoplus (1-p_s) \bullet (\mathbf{v}_0, \mathbf{q})] \sim_i (\mathbf{v}, \mathbf{p})$.

Recall, any rule ϕ induces the preference $\succsim_i$, over the class of independent combination of games of bankruptcy problems, for the risk neutral individual i who has von Neumann–Morgenstern preferences over monetary lotteries. For any bankruptcy problem $(\mathbf{c}, E, \mathbf{p})$, let $\mathbf{v}$ and $\mathbf{v}_0$ denote the situations $(\mathbf{c}, E)$ and $(\mathbf{c}, 0)$, respectively. The independent combination, denoted by $\bigotimes_{s\in S}[p_s \bullet (\mathbf{v}, \boldsymbol{\delta}_s) \bigoplus (1-p_s) \bullet (\mathbf{v}_0, \mathbf{q})]$, is just the *independent* conducted of the lottery of "ex–post" problems $(\mathbf{v}, \boldsymbol{\delta}_s)$ and $(\mathbf{v}_0, \mathbf{q})] \sim_i (\mathbf{v}, \mathbf{p})$ with probabilities p_s and $1-p_s$. Thus, each of the "ex–post" problems has a probability of realization equal to p_s which is the probability assessed for the realization of the state s. Thus, an individual perhaps may percieve of the corresponding independent combination of gambles as presenting more opportunities than there would be in the original problem $(\mathbf{v}, \mathbf{p})$. This may be justified by the observation that, while in the original problem the realization of any one state excludes the realization of the others, with independence the manifestation each of the "ex–post" problem is independent of the manifestation of the others. However, these are the only opportunities, provided under the independent combination of gambles, as everything else is just the corresponding zero problem. The question arises to what degree does the rule substantiate this perception of such individuals. The axiom says the the rule makes such an individual indifferent between the two possibilities. As shall be seen, this axiom characterizes rules that have the ex–post form. For this to be the key, the notion of corresponding *zero problems* is essential as they act as "absorbers" of remaining probabilities in the sense that, no matter what there likelihood of occurrence, they always result in a zero share of every individual under any rule. We emphasize the last point as this feature is not present in every setting where ex–post forms seek a characterization. Bankruptcy problems do present such an instance where the feature exists.

7. Main Results

We present our main results below. All the proofs are relegated to the appendix. We begin with Theorem 1 which states that the class of parametric rules Φ, described in section 4, satisfy the three axioms: *claim monotonicity*, *weak consistency* and *no reward for more irrelevant claims.* As section 4 already makes a case that the class Φ is very large, the point of Theorem 1 is to indicate that the scope of applicability of Theorem 2 is at least as large as the class Φ.

Theorem 1. *Any rule $\phi \in \Phi$ satisfies claim monotonicity, weak consistency and no reward for more irrelevant claims.*

Within the class of rules that satisfy the axioms of *claim monotonicity*, *weak consistency* and *no reward for more irrelevant claims*, Theorem 2 is the main result that characterizes those rules that have the ex–ante form. Since the class Φ is a sublass of such rules according to Theorem 1, we conclude that the characterization achieved by Theorem 2 pins down the critical feature, centering around the way rules process the inherent riskiness that manifest in the state contingent claims of the individuals, that ex–ante rules within this class have.

Theorem 2. *Consider any rule that satisfies claim monotonicity, weak consistency and no reward for more irrelevant claims. The rule satisfies no penalty for risk and no sudden response to uncertainty, if and only if, it is an ex–ante rule.*

In the light of Theorem 1 and Theorem 2, the following corollary is immediate.

Corollary 1. *Any rule $\phi \in \Phi$ has the ex–ante form, if and only if, ϕ satisfies no penalty for risk and no sudden response to uncertainty.*

The key to the proof of Theorem 2 is the Lemma 1 which is stated next. For the statement of the following lemma, we now define the notion of the "equivalent deterministic profile". Define the map, $(\mathbf{c}, \mathbf{p}) \in \mathscr{C} \times \Delta(S) \mapsto \mathbf{c}^*(\mathbf{c}, \mathbf{p}) \in \mathscr{C}$ as $\mathbf{c}^*(\mathbf{c}, \mathbf{p}) := \langle \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S : i \in N \rangle$. Then $\mathbf{c}^*(\mathbf{c}, \mathbf{p})$ is the *equivalent deterministic profile* of $\mathbf{c}$ at the assessment of state probabilities $\mathbf{p} \in \Delta(S)$. Lemma 1 describes the properties of the equivalent deterministic profile which are used in the proof of Theorem 2.

Lemma 1. *If a rule ϕ satifies no penalty for risk, claim monotonicity, weak consistency and no reward for more irrelevant claims, then for any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ such that $E \leq \min_{i \in N} \bar{c}_i(\mathbf{p})$ each of the following hold:*

1. $(\mathbf{c}^*(\mathbf{c}, \mathbf{p}), E, \mathbf{p}) \in \mathscr{D}$.

2. $\phi(\mathbf{c}, E, \mathbf{p}) = \phi(\mathbf{c}^*(\mathbf{c}, \mathbf{p}), E, \mathbf{p})$.

Having described our characterization theorem for the ex–ante form, now we proceed to describe Theorem 3 which characterizes the ex–post form of rules. Since the characterization is based on considerations of the way a rule makes an individual perceive different configurations consisting of bankruptcy problems, we bring to fore these considerations before the statement of Theorem 2. Recall, the class of all independent combinations of gambles has been denoted by $\mathscr{I}_{\mathscr{D}}$. For each individual i, driven by von Neumann–Morgenstern preferences over money lotteries, $\succsim_i$ denotes the induced preference over $\mathscr{I}_{\mathscr{D}}$.[5]

Theorem 3. *The rule ϕ has the ex–post form, if and only if, the induced $\succsim_i$ satisfies indifference to independent combinations for every risk neutral $i \in N$.*

Remark. Here we point out that the theorem above is not a mere statement of definitional equivalence of two notions: "ex–post rule" and "Indifference to Independent Combinations". A rule is required to make an individual indifferent between the original problem and its corresponding "independent combination of gambles" version *only if* that individual is risk neutral. The rule is silent about how an individual compares these two versions of a problem if he is *not* risk neutral. Further, it is *not* required that at least one or more individuals are risk neutral.

The next result offers a partial comparative static between the ex–ante and the ex–post forms of a rule induced by some $h \in \mathscr{H}$. Recall, from the section 5, many rules such as the Talmud rule, the Minimal Overlap rule and any member of the class of Priority Augmented Weighted Constrained Equal Awards rule admit natural ex–ante and ex–post versions through this mechanism. The theorem below says, for all low enough levels of the resource, if all but one individuals have deterministic claims, then the ex–ante rule defined by h awards a larger share to the odd individual than does the ex–post rule defined by h. In this sense, perhaps when the planner proposes a choice between the ex–ante and the ex–post versions of one of the standard rules, it is possible that individuals may be more inclined to accept the ex–post version.

Theorem 4. *Let ϕ^{EP} and ϕ^{EA} be the ex–post and ex–ante rules defined by some $h \in \mathscr{H}$. Consider any individual $i \in N$, and any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ such that $E \leq \min_{j \in N} \bar{c}_j(\mathbf{p})$ and $\mathbf{c}_j$ is deterministic for every $j \in N \setminus \{i\}$. Then $\phi_i^{EA}(\mathbf{c}, E, \mathbf{p}) \geq \phi_i^{EP}(\mathbf{c}, E, \mathbf{p})$. For $E > \min_{s \in S} c_{is}$, given a profile of claims, if $\mathbf{p} \in \Delta^{\circ}(S)$ and $\mathbf{c}_i$ non–deterministic, then the inequality is strict.*[6]

[5]See the last paragraph of section 2 for the complete definition of $\succsim_i$.

[6]For any subset K, of a topological space $(X, \mathscr{T}_X)$, K° denotes the *interior* of K.

Theorems 2 and 3 provide a very top–level characterization of the ex–ante and the ex–post forms of rules to resolve bankruptcy problems. One way to think of the relevance of these theorems is that whenever a rule — or a class of rules — from the standard bankruptcy literature in the "deterministic" setting, is considered for extension to the setting involving "uncertainty", then standard characterizations of the corresponding "deterministic" versions are adaptable to characterizations of the corresponding "ex–ante" and "ex–post" forms. The key idea is whether the rule being proposed should be chosen so as to satisfy either the property of *no penalty for risk* or that of *indifference to independence combinations*. Within the class Φ, the two choices are not compatible.

8. Geometry of the characterization of the ex–ante form

To demonstrate geometrically the characterization argument of the ex–ante form, consider a set up with three individuals and four states of nature. Since any profile of state contingent claims is a tuple consisting of one profile of individual claims for each state, it is possible to represent such a profile of state contingent claims as a collection of points in the first orthant of an Euclidean space with dimension equal to the number of individuals. Each point of the collection represents the profile of individual claims corresponding to one state. Thus, for a typical profile of state contingent claims, the collection has points equal in number to that of the states. However, given a profile of state contingent claims, it is possible for some of these points to coincide. For any two states, the points representing them coincide precisely when the profile of individual claims in the two states are identical; *i.e.*, any individual's claims are the same in both the states. With these comments in place, consider FIGURE 1.

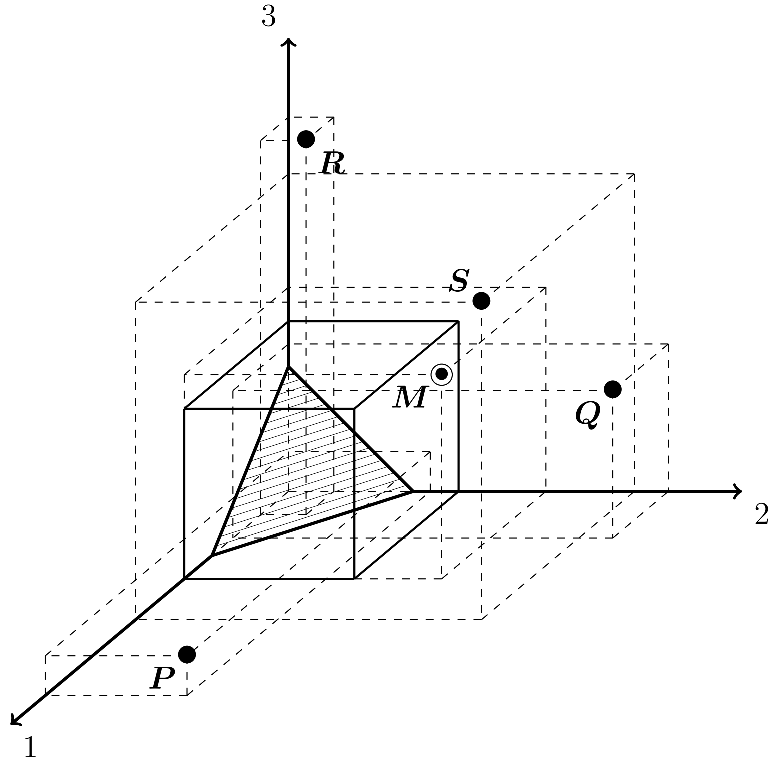
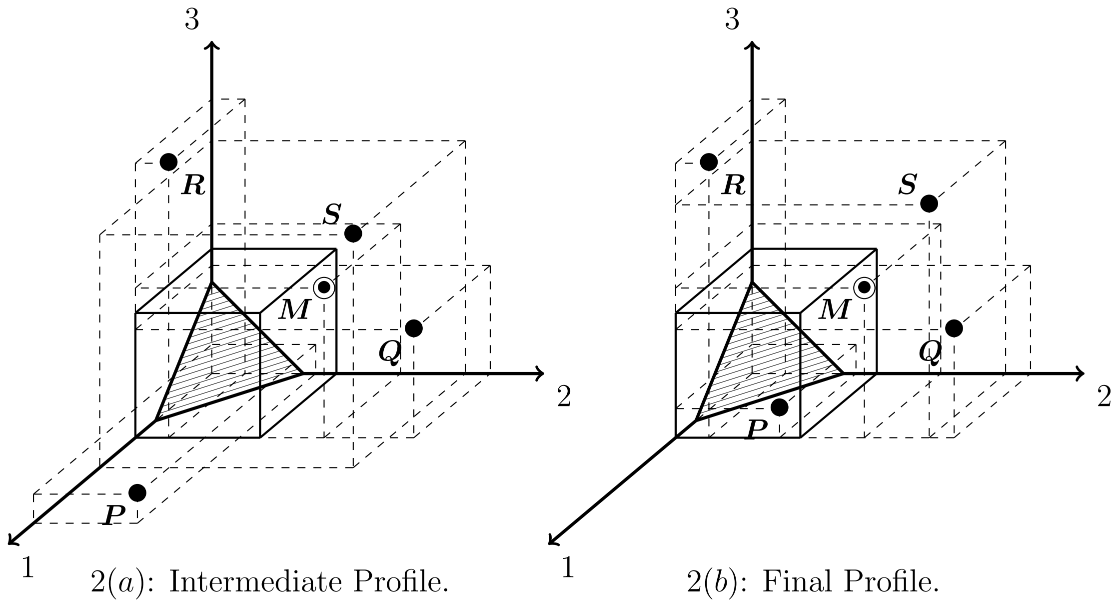


FIGURE 1: A BANKRUPTCY PROBLEM — The axes labelled 1,2 and 3 correspond to three individuals. The four dots labelled P, Q, R and S correspond to the state-wise profile of individual claims for four states of nature. The circled dot M indicates the vector of individual claims given the four state-wise profile of individual claims and an assessment of state probabilities $\mathbf{p}$. The corner of the square shaped box with dark edges which is farthest from the origin is the vector all whose coordinates equal the smallest component of the vector of expected individual claims M. The shaded triangular surface is the resource level E because any point on it has the sum of coordinates as E.

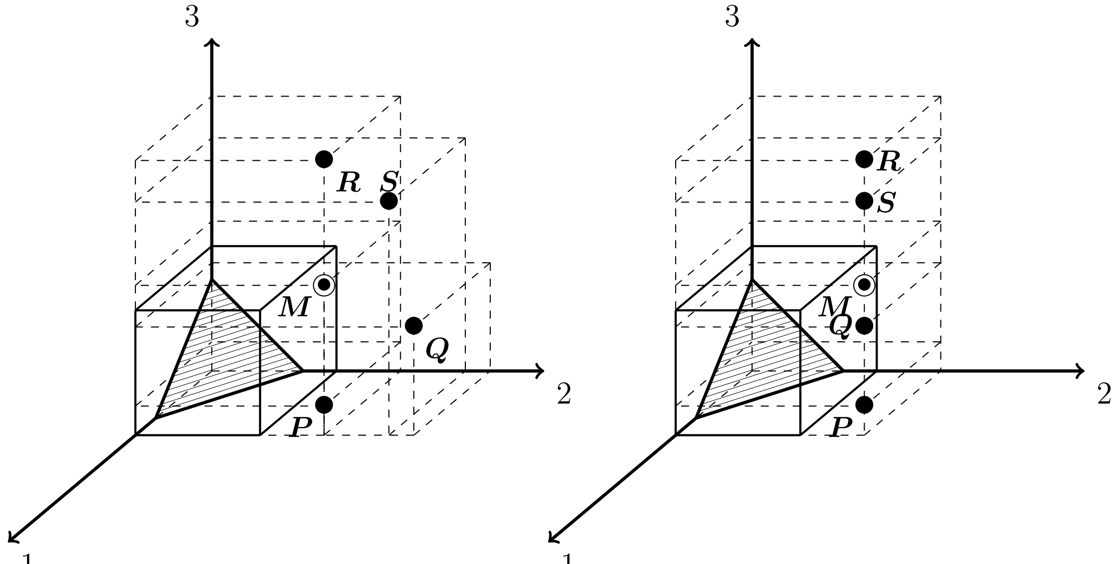
Since the demonstration considers four states of nature, the figure consists of four points labelled P, Q, R and S with each point representing one such state. The points are considered in the first orthant of a three dimensional Euclidean space since we have three individuals. The axes are labelled 1, 2 and 3. The interpretation for any such point is that the orthogonal component of the point along the axis corresponding to any individual is the individual's claim in that state. For any assessment of state probabilities, the vector of expected claims of the individuals is a point in the convex hull of the state-wise profile of claims. This is illustrated as the point labelled M. The shaded triangular surface represents those points of the first orthant whose sum of coordinates is resource, say E. For the given profile of state contingent claims and the resource, together with the fixed assessment of state probabilities, to constitute a bankruptcy problem, it must be the case that the state-wise sum of individual claims is at least as much as the resource. That this is indeed the case is ensured by positioning each of the four points "above" the triangular surface. For purposes of this demonstration, we consider the resource to be at most the smallest expected claim across individuals. This is captured by having the triangular surface enclosed in the cube whose edges are shown with dark lines. Therefore, the vector of individuals' expected claims is on the extended face of the cube which is parallel to the plane defined by axes 2 and 3. Now, consider Figure 2.



2(a): Intermediate Profile. 2(b): Final Profile.

Figure 2: First individual processed — The first coordinates of Q and R are increased to that of M to obtain figure 2(a) from figure 1. Next, the first coordinates of P and S are decreased to that of M to obtain figure 2(b) from figure 2(a). Thus, in figure 2(b), each of P, Q, R, S and M are in the same plane which is parallel to the plane defined by the axes labelled 2 and 3.

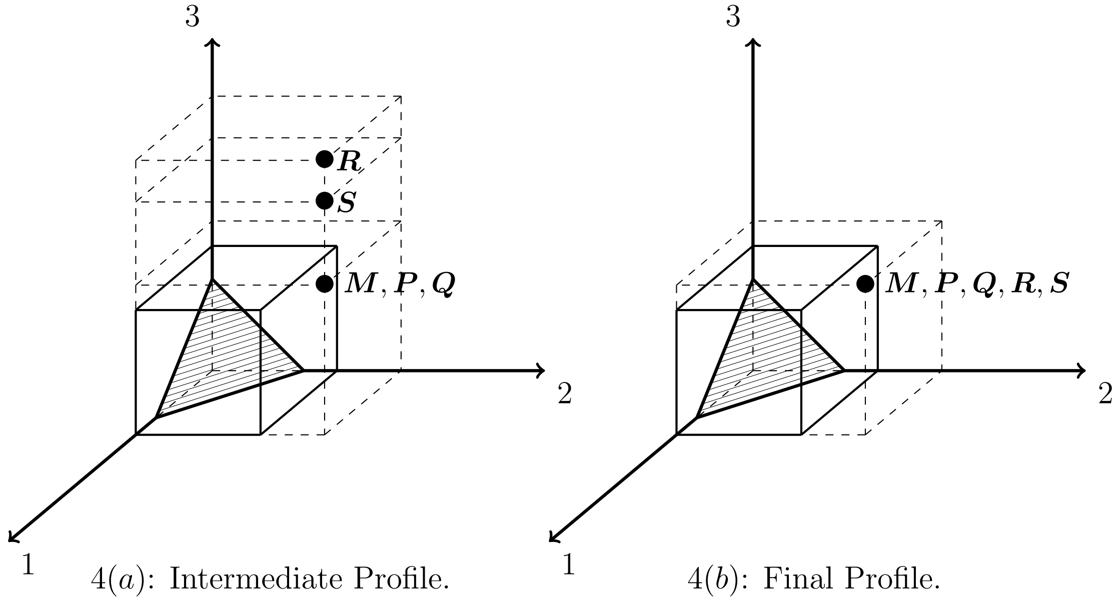
The profile shown in Figure 2(a) differs from Figure 1 in that the first coordinates of the points Q and R are increased to that of M. Since the claims of individuals 2 and 3 for every state is unchanged, by *claims monotonicity*, individual 1 gets at least as much in the profile of Figure 2(a) as she does in the profile of Figure 1. The profile shown in Figure 2(b) differs from that of Figure 2(a) in that the first coordinates of the points P and S are decreased to that of M. The state-wise claim of individual 1 is at least as much as the resource in both the profiles of Figure 2(a) and Figure 2(b). By *no reward for more irrelevant claims*, individual 1 gets at most as much in the profile of Figure 2(a) as she does in the profile of Figure 2(b). Thus, individual 1 gets at least as much in the profile shown in Figure 2(b) as she does in the profile shown in Figure 1. Notice, from the perspective of individual 1, the profile in Figure 2(b) is riskless and has the same mean as the profile in Figure 1 which is shown to be risky. Since the state-wise claims of individuals 2 and 3 are the same across the two profiles, by *no penalty for risk*, individual 1 gets at least as much in the profile of Figure 1 as she does in the profile of Figure 2(b). Together with the earlier conclusion, the individual is seen to obtain the same across both the profiles of Figure 1 and Figure 2(b). By *weak consistency*, individuals 2 and 3 also get the same across the two profiles. Thus, the rule treats both the profiles shown in Figure 1 and Figure 2(b) identically. Now, consider Figure 3.



3(a): Intermediate Profile. 3(b): Final Profile.

FIGURE 3: SECOND INDIVIDUAL PROCESSED — The second coordinates of P and R are increased to that of M to obtain figure 3(a) from figure 2(b). Next, the second coordinates of Q and S are decreased to that of M to obtain figure 3(b) from figure 3(a). Thus, in figure 3(b), each of P, Q, R, S and M are in the same line which is normal to the plane defined by the axes labelled 1 and 2.

The profile shown in FIGURE 3(a) differs from FIGURE 2(b) in that the second coordinates of the points P and R are increased to that of M. Since the claims of individuals 1 and 3 for every state is unchanged, by *claims monotonicity*, individual 2 gets at least as much in the profile of FIGURE 3(a) as she does in the profile of FIGURE 2(b). The profile shown in FIGURE 3(b) differs from that of FIGURE 3(a) in that the second coordinates of the points Q and S are decreased to that of M. The state-wise claim of individual 2 is at least as much as the resource in both the profiles of FIGURE 3(a) and FIGURE 3(b). By *no reward for more irrelevant claims*, individual 2 gets at most as much in the profile of FIGURE 3(a) as she does in the profile of FIGURE 3(b). Thus, individual 2 gets at least as much in the profile shown in FIGURE 3(b) as she does in the profile shown in FIGURE 2(b). Notice, from the perspective of individual 2, the profile in FIGURE 3(b) is riskless and has the same mean as the profile in FIGURE 2(b) which is shown to be risky. Since the state-wise claims of individuals 1 and 3 are the same across the two profiles, by *no penalty for risk*, individual 2 gets at least as much in the profile of FIGURE 2(b) as she does in the profile of FIGURE 3(b). Together with the earlier conclusion, the individual is seen to obtain the same across both the profiles of FIGURE 2(b) and FIGURE 3(b). By *weak consistency*, individuals 1 and 3 also get the same across the two profiles. Thus, the rule treats both the profiles shown in FIGURE 2(b) and FIGURE 3(b) identically. Now, consider FIGURE 4.



4(a): Intermediate Profile. 4(b): Final Profile.

FIGURE 4: THIRD INDIVIDUAL PROCESSED — The third coordinates of P and Q are increased to that of M to obtain figure 4(a) from figure 3(b). Next, the third coordinates of R and S are decreased to that of M to obtain figure 4(b) from figure 4(a). Thus, in figure 4(b), each of P, Q, R, and S coincide with M.

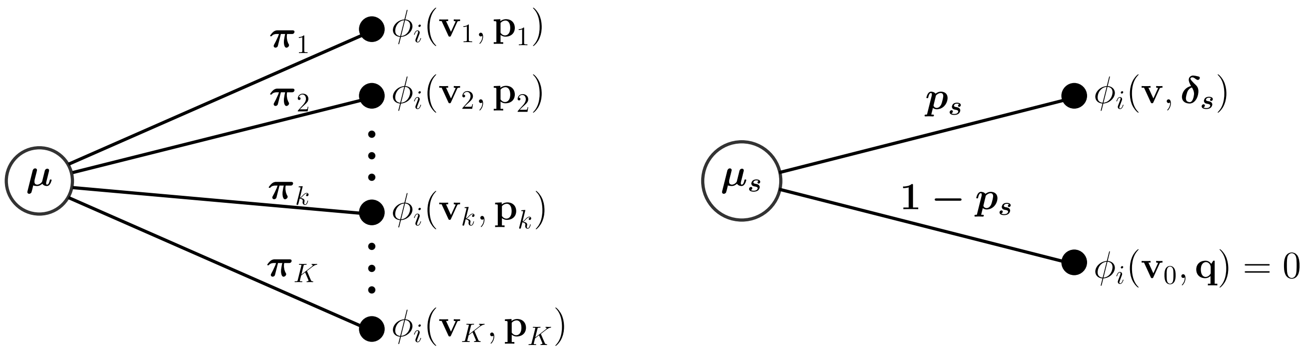
The profile shown in FIGURE 4(a) differs from FIGURE 3(b) in that the third coordinates of the points P and Q are increased to that of M. Since the claims of individuals 1 and 2 for every state is unchanged, by *claims monotonicity*, individual 3 gets at least as much in the profile of FIGURE 4(a) as she does in the profile of FIGURE 3(b). The profile shown in FIGURE 4(b) differs from that of FIGURE 4(a) in that the third coordinates of the points R and S are decreased to that of M. The state-wise claim of individual 3 is at least as much as the resource in both the profiles of FIGURE 4(a) and FIGURE 4(b). By *no reward for more irrelevant claims*, individual 3 gets at most as much in the profile of FIGURE 4(a) as she does in the profile of FIGURE 4(b). Thus, individual 3 gets at least as much in the profile shown in FIGURE 4(b) as she does in the profile shown in FIGURE 4(a). Notice, from the perspective of individual 3, the profile in FIGURE 4(b) is riskless and has the same mean as the profile in FIGURE 3(b) which is shown to be risky. Since the state-wise claims of individuals 1 and 2 are the same across the two profiles, by *no penalty for risk*, individual 3 gets at least as much in the profile of FIGURE 3(b) as she does in the profile of FIGURE 4(b). Together with the earlier conclusion, the individual is seen to obtain the same across both the profiles of FIGURE 3(b) and FIGURE 4(b). By *weak consistency*, individuals 1 and 2 also get the same across the two profiles. Thus, the rule treats both the profiles shown in FIGURE 3(b) and FIGURE 4(b) identically.

From the concluding statements of the last three paragraphs, it follows that the rule treats the profiles shown in FIGURE 1 and FIGURE 4(b) identically. Thus, it is concluded that the rule processes any bankruptcy problem through the resulting vector of expected claims of the individuals.

This is the content of Lemma 1.

9. Strategy of the characterization of the ex–post form

The characterization of the ex–post form is based on the way a chosen rule, to resolve bankruptcy problems, makes a certain class of individuals perceive different situations that they may find themselves in which are different configurations built out of several bankruptcy problems. In particular, we envisage the individuals to be driven by von Neumann–Morgenstern preferences over money lotteries. Now, fixing a rule under consideration, say ϕ, let $(\mathbf{c}, E, \mathbf{p})$ be a bankruptcy problem. Recall, the resulting pair $(\mathbf{v}, E)$ is denoted by $\mathbf{v}$ and is called a *situation*. Also, we have a corresponding *zero situation*, denoted by $\mathbf{v}_0$, which is the pair $(\mathbf{c}, 0)$. That is, for the situation $\mathbf{v}$, the correponding zero situation is obtained by changing the level of the resource to zero. Since a rule to divide the resource never awards negative shares to any individual, when the level of the resource is zero it must award every individual the share zero. This feature of bankruptcy problems is critical in identifying the notion of zero situation as then, for every individual i, we know that $\phi_i(\mathbf{v}_0, \mathbf{q})$ for any assessment of state probabilities $\mathbf{q} \in \Delta(S)$. As will be seen, it is the existence of this natural notion of zero situation corresponding to any situation is what makes our characterization of the ex–post possible with such generality.



5(a) A general gamble. 5(b) A special gamble.

FIGURE 5: GAMBLE OVER BANKRUPTCY PROBLEMS — Using a randomization device having K mutually exclusive and exhaustive outcomes, with probabilities $\pi_1, \ldots, \pi_K$, the decision makers are made to engage in the kth. bankruptcy problem $(\mathbf{v}_k, \mathbf{p}_k)$ inducing a monetary outcome of $\phi_i(\mathbf{v}_k, \mathbf{p}_k)$, for individual i, with probability π_k. The monetary outcomes are indicated to the right of the terminal nodes. This is a general "gamble over bankruptcy problems" as illustrated in part (a). Part (b) illustrates a very special gamble. Starting with bankruptcy problem $(\mathbf{v}, \mathbf{p})$, two new bankruptcy problems are derived which are $(\mathbf{v}, \boldsymbol{\delta_s})$ and the corresponding zero problem $(\mathbf{v}_0, \mathbf{q})$. These two problems are played with probabilities p_s and $1 - p_s$, respectively. Note, the zero problem gives individual i a monetary return of zero as the resource in the situation $\mathbf{v}_0$ is zero.

Now, consider FIGURE 5. Demonstrated in FIGURE 5(a) is a *gamble over bankruptcy problems.*

What is meant is that via some randomization device, which offers K possible mutually exclusive and exhaustive outcomes with repsective probabilities $\pi_1, \pi_2, \ldots, \pi_K$, the bankruptcy problem $(\mathbf{v}_k, \mathbf{p}_k) \in \mathscr{D}$ is realized for $k \in \{1, 2, \ldots, K\}$. The gamble is formally denoted by $[\bigoplus_{k=1}^{K} \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)]$. As a result, the realization of the problem $(\mathbf{v}_k, \mathbf{p}_k)$ implies a monetary reward of $\phi_i(\mathbf{v}_k, \mathbf{p}_k)$ for individual i. Since the individual envisaged is driven by von Neumann–Morgenstern preference over monetary outcomes, if individual is risk neutral then she perceives this gamble via its impied expected wealth, namely $\sum_{k=1}^{K} \pi_k \cdot \phi_i(\mathbf{v}_k, \mathbf{p}_k)$. Now, FIGURE 5($b$) presents a very specific gamble. Given the bankruptcy problem $(\mathbf{v}, \mathbf{p})$ and any state $s \in S$, this gamble results in the problem $(\mathbf{v}, \boldsymbol{\delta_s})$ with probability p_s and the problem $(\mathbf{v}_0, \mathbf{q})$ with probability $1 - p_s$. As we have already seen that $\phi_i(\mathbf{v}_0, \mathbf{q})$, we conclude that this special gamble presents individual i with expected wealth $p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta_s})$.

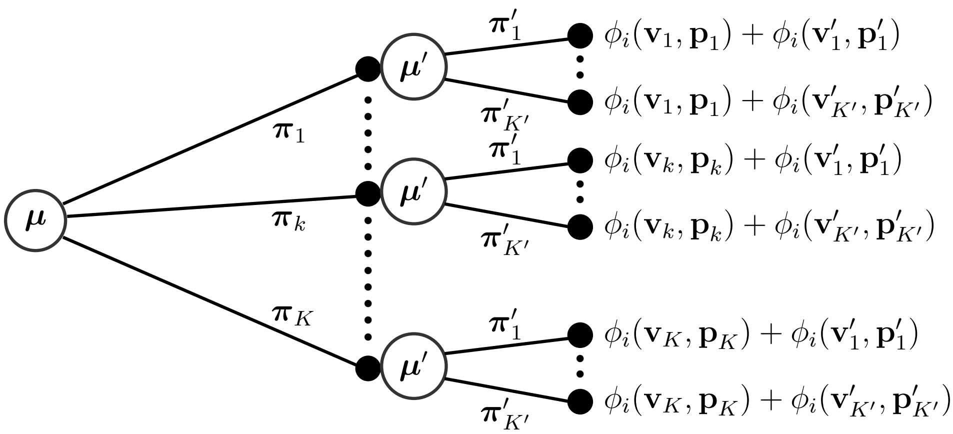


FIGURE 6: INDEPENDENT COMBINATION OF GAMBLES — Two gambles, μ and μ' are conducted *independently.* Thus, the subtree rooted at any any outcome of the gamble μ is not dependent on the resulting outcome. The terminal nodes represent the outcomes of the independent combination of the two gambles. The money receieved by individual i is indicated to the right of each terminal node. The probability of the realization of any terminal node is the product of the probabilities indicated along the edges leading to that terminal node from the root of the tree.

Now, consider FIGURE 6. Demontrated is an *independent combination of gambles* starting with two gambles over bankruptcy problems. The first gamble is denoted by μ which is $[\bigoplus_{k=1}^{K} \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)]$. That is, there are K bankruptcy problems, with the kth. problem having a probability of realization π_k. The second gamble is denoted by μ' which is $[\bigoplus_{k=1}^{K'} \pi'_k \bullet (\mathbf{v}'_k, \mathbf{p}'_k)]$. That is, there are K' bankruptcy problems, with the kth. problem having a probability of realization π'_k. Since *independence* is in the sense of probability theory, no matter what the outcome of the first gamble is, the gamble following is the same. This independent combination of gambles is formally denoted by $[\bigoplus_{k=1}^{K} \pi_k \bullet (\mathbf{v}_k, \mathbf{p}_k)] \bigotimes [\bigoplus_{k=1}^{K'} \pi'_k \bullet (\mathbf{v}'_k, \mathbf{p}'_k)]$. Since for any $k \in \{1, 2, \ldots, K\}$ and $k' \in \{1, 2, \ldots, K'\}$ the terminal node is realized with probability $\pi_k \cdot \pi_{k'}$, the expected wealth of individual i from this independent combination of gambles is the weighted sum of terms of the form $\phi_i(\mathbf{v}_k, \mathbf{p}_k) + \phi_i(\mathbf{v}'_k, \mathbf{p}'_k)$ with respective weights being $\pi_k \cdot \pi_{k'}$. In this figure, only the independent combination of two gambles has been illustrated. The notion of "independent combinations of gambles" stand defined analogously for arbitrarily many finite number of gambles over bankruptcy problems. The number of gambles, in any independent combination, corresponds to the depth of the tree.

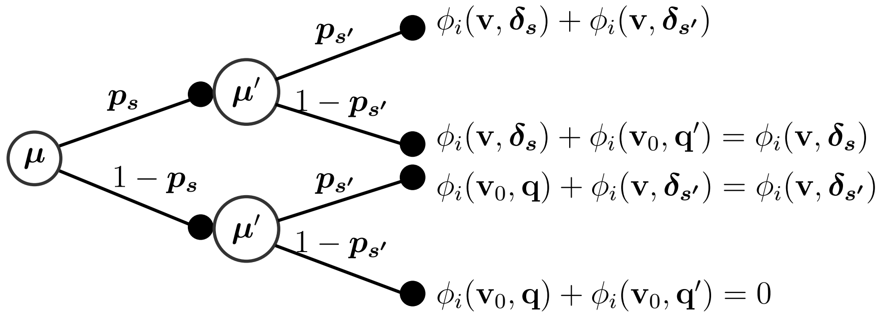


FIGURE 7: THE SPECIAL COMBINATION — For the state space to consist only of two states s and s', for some bankruptcy problem $(\mathbf{v}, \mathbf{p})$, the independent combination of independent gambles presented here has the feature that the monetary outcomes $\phi_i(\mathbf{v}, \boldsymbol{\delta_s})$ and $\phi_i(\mathbf{v}, \boldsymbol{\delta_{s'}})$ have *independent* probabilities of realization as p_s and $p_{s'}$, respectively. Notice, this is the case precisely because the corresponding zero situation $\mathbf{v}_0$ is such that, for any $\mathbf{q}, \mathbf{q}' \in \Delta(S)$, $\phi_i(\mathbf{v}_0, \mathbf{q})$ and $\phi_i(\mathbf{v}_0, \mathbf{q}')$ are both zero. This is so as $\mathbf{v}_0$ is obtained from $\mathbf{v}$ by setting the resource to zero.

Finally, consider FIGURE 7. For exactly two states of nature s and s', and any bankruptcy problem $(\mathbf{v}), \mathbf{p}$, two gambles are obtained corresponding to each of the two "ex–post" problem. That is, the first gamble randomzises between $(\mathbf{v}, \boldsymbol{\delta_s})$ and the corresponding zero problem $(\mathbf{v}_0), \mathbf{q}$ with probabilities p_s and $1 - p_s$, respectively. The second gamble randomzises between $(\mathbf{v}, \boldsymbol{\delta_{s'}})$ and the corresponding zero problem $(\mathbf{v}_0), \mathbf{q}'$ with probabilities $p_{s'}$ and $1 - p_{s'}$, respectively. Since the zero problems result in zero share for the individual i, we conclude that the realized monetary rewards for individual i are $\phi_i(\mathbf{v}, \boldsymbol{\delta_s}) + \phi_i(\mathbf{v}, \boldsymbol{\delta_{s'}})$, $\phi_i(\mathbf{v}, \boldsymbol{\delta_s})$ and $\phi_i(\mathbf{v}, \boldsymbol{\delta_{s'}})$ (the remaining probability being on the reward of zero) with probabilities $p_s \cdot p_{s'}$, $p_s \cdot (1 - p_{s'})$ and $(1 - p_s) \cdot p_{s'}$, respectively. The expression for the implied expected wealth then simplifies to $p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta_s}) + p_{s'} \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta_{s'}})$. Further, the original problem $(\mathbf{v}, \boldsymbol{\delta_s})$ implies a wealth $\phi_i(\mathbf{v}, \boldsymbol{\delta_s})$ for this individual. Thus, the rule ϕ has the ex–post form, if and only if, the risk neutral individual i is *indifferent* between the original problem and the independent combination of gambles as described in the figure. This is the content of Theorem 3.

10. Logical Independence of Axioms

Given our characterization results, namely, Theorem 2 and Theorem 3, we shall provide two examples of rules in this section. The first example satisfies the axioms *no penalty for risk*, *claim* *monotonicity*, *weak consistency* and *no reward for more irrelevant claims* but violates the property of *no sudden response to uncertainty.* The second example satisfies the axioms *no sudden response to uncertainty*, *claim monotonicity*, *weak consistency* and *no reward for more irrelevant claims* but violates the property of *no penalty for risk.*

Example 1. Define the map $\phi^* : \mathscr{D} \to \mathbb{R}^N_+$ as follows. Fix any problem $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. For every $i \in N$, let $\phi^*_i(\mathbf{c}, E, \mathbf{p}) := \min\{\|(\mathbf{c}_i, \mathbf{p})\|_2, \lambda\}$. Here, λ satisfies $\sum_{i \in N} \min\{\|(\mathbf{c}_i, \mathbf{p})\|_2, \lambda\} = E$. Since $\sum_{i\in N} \|(\mathbf{c}_i, \mathbf{p})\|_2 \geq \sum_{i \in N} \|(\mathbf{c}_i, \mathbf{p})\|_1 = \sum_{s \in S}[p_s \cdot \sum_{i\in N} c_{is}]$, it follows that $\sum_{i \in N} \|(\mathbf{c}_i, \mathbf{p})\|_2 \geq E$. Thus, such a λ indeed exists. λ is unique if the inequality is strict. If the inequality is indeed an equality, then all solutions define the same value of the map at the problem. Thus, the rule is indeed well–defined. Observe, the map ϕ^* is continuous in E. Hence, ϕ^* is indeed a rule according to our definition in section 2.

Consider any $\mathbf{c}'_i \geq \mathbf{c}_i$. Let λ' satisfy $\min\{\|(\mathbf{c}'_i, \mathbf{p})\|_2, \lambda'\} + \sum_{j \in N\setminus\{i\}} \min\{\|(\mathbf{c}_i, \mathbf{p})\|_2, \lambda'\} = E$. Since $\mathbf{c}'_i \geq \mathbf{c}_i$, we have $\|(\mathbf{c}'_i, \mathbf{p})\|_2 \geq \|(\mathbf{c}_i, \mathbf{p})\|_2$. If $\min\{\|(\mathbf{c}'_i, \mathbf{p})\|_2, \lambda'\} < \min\{\|(\mathbf{c}_i, \mathbf{p})\|_2, \lambda\}$, then $\lambda' < \lambda$. Hence, for every $j \in N \setminus \{i\}$, it follows that $\min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda'\} < \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda\}$. Hence, we have: $\min\{\|(\mathbf{c}'_i, \mathbf{p})\|_2, \lambda'\} + \sum_{j\in N\setminus\{i\}} \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda'\} < \sum_{j \in N} \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda\}$. By definition of λ' and λ both sides of the above inequality respectively equal E. Since this is a contradiction, our supposition has to be wrong. That is, ϕ^* satisfies *claim monotonicity.*

Consider any $\mathbf{c}'_i$ such that $\phi^*_i((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) = \phi^*_i(\mathbf{c}, E, \mathbf{p})$. Let λ' satisfy $\min\{\|(\mathbf{c}'_i, \mathbf{p})\|_2, \lambda'\} + \sum_{j\in N\setminus\{i\}} \min\{\|(\mathbf{c}_i, \mathbf{p})\|_2, \lambda'\} = E$. Assume, without loss of generality, $\lambda' \geq \lambda$. Hence, for every $j \in N \setminus \{i\}$, $\min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda'\} \geq \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda\}$. Since $\min\{\|(\mathbf{c}'_i, \mathbf{p})\|_2, \lambda'\} = \min\{\|(\mathbf{c}_i, \mathbf{p})\|_2, \lambda\}$, we obtain $\min\{\|(\mathbf{c}'_i, \mathbf{p})\|_2, \lambda'\} + \sum_{j\in N\setminus\{i\}} \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda'\} \geq \sum_{j\in N} \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda\}$. By definition of λ' and λ, respectively, both sides of the last inequality equal E. Hence, to avoid a contradiction, we must have: $\min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda'\} = \min\{\|(\mathbf{c}_j, \mathbf{p})\|_2, \lambda\}$ for every $j \in N \setminus \{i\}$. From the definition of the rule ϕ^*, we have: $\phi^*_j((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) = \phi^*_j(\mathbf{c}, E, \mathbf{p})$ for every $j \in N \setminus \{i\}$. That is, the rule ϕ^* satisfies *weak consistency.*

Observe, $\|(\mathbf{c}_i, \mathbf{p})\|_2 \geq \min_{s\in S} c_{is}$. Hence, $\min_{s \in S} c_{is} \geq E$ implies $\|(\mathbf{c}_i, \mathbf{p})\|_2 \geq E$. *No reward for more irrelevant claims* follows now by the definition of ϕ^*. Since $\|(\mathbf{x}, \mathbf{p})\|_2 \geq \|(\mathbf{x}, \mathbf{p})\|_1$ for every $\mathbf{x} \in \mathbb{R}^S_+$ and $\mathbf{p} \in \Delta(S)$, it follows that *no penalty for risk* is true for the rule ϕ^*. In fact, note that $\|(\mathbf{c}_i, \mathbf{p})\|_2^2 = \sigma^2(\mathbf{c}_i, \mathbf{p}) + \|(\mathbf{c}_i, \mathbf{p})\|_1^2$, where $\sigma^2(\mathbf{c}_i, \mathbf{p}) := \sum_{s\in S} p_s \cdot (c_{is} - \bar{c}_i(\mathbf{p}))^2$. Thus, it is not possible that ϕ^* is a rule that is of the ex–ante form. Since ϕ^* satisfies all the four axioms discussed, by Theorem 2, it must violate *no sudden response to uncertainty.* The presentation of this example is complete. ■

Example 2. Define the map $\phi^{**} : \mathscr{D} \to \mathbb{R}^N_+$ as follows. Fix an arbitrary state $s_0 \in S$. Fix any problem $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. For every $i \in N$, let $\phi^{**}_i(\mathbf{c}, E, \mathbf{p}) := \min\{c_{is_0}, \lambda\}$. Here, λ satisfies $\sum_{i \in N} \min\{c_{is_0}, \lambda\} = E$. Since $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, it follows that $\sum_{i \in N} c_{is_0} \geq E$. Thus, such a λ indeed exists. λ is unique if the inequality is strict. If the inequality is indeed an equality, then all solutions define the same value of the map at the problem. Thus, the rule is indeed well–defined. Observe, the map ϕ^{**} is continuous in E. Hence, ϕ^{**} is indeed a rule according to our definition in section 2.

Consider any $\mathbf{c}'_i \geq \mathbf{c}_i$. Let λ' satisfy $\min\{c'_{is_0}, \lambda'\} + \sum_{j \in N \setminus \{i\}} \min\{c_{js_0}, \lambda'\} = E$. Since $\mathbf{c}'_i \geq \mathbf{c}_i$, we have $c'_{is_0} \geq c_{is_0}$. If $\min\{c'_{is_0}, \lambda'\} < \min\{c_{is_0}, \lambda\}$, then $\lambda' < \lambda$. Hence, for every $j \in N \setminus \{i\}$, it follows that $\min\{c'_{js_0}, \lambda'\} < \min\{c_{js_0}, \lambda\}$. Hence, we have: $\min\{c'_{is_0}, \lambda'\} + \sum_{j \in N \setminus \{i\}} \min\{c_{js_0}, \lambda'\} < \sum_{j \in N} \min\{c_{js_0}, \lambda\}$. By definition of λ' and λ both sides of the above inequality respectively equal E. Since this is a contradiction, our supposition has to be wrong. That is, ϕ^{**} satisfies *claim monotonicity*.

Consider any $\mathbf{c}'_i$ such that $\phi^{**}_i((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) = \phi^{**}_i(\mathbf{c}, E, \mathbf{p})$. Let λ' satisfy $\min\{c'_{is_0}, \lambda'\} + \sum_{j \in N \setminus \{i\}} \min\{c_{js_0}, \lambda'\} = E$. Assume, without loss of generality, $\lambda' \geq \lambda$. Hence, for every $j \in N \setminus \{i\}$, $\min\{c_{js_0}, \lambda'\} \geq \min\{c_{js_0}, \lambda\}$. Since $\min\{c'_{is_0}, \lambda'\} = \min\{c_{is_0}, \lambda\}$, we obtain $\min\{c'_{is_0}, \lambda'\} + \sum_{j \in N \setminus \{i\}} \min\{c_{js_0}, \lambda'\} \geq \sum_{j \in N} \min\{c_{js_0}, \lambda\}$. By definition of λ' and λ, respectively, both sides of the last inequality equal E. Hence, to avoid a contradiction, we must have: $\min\{c_{js_0}, \lambda'\} = \min\{c_{js_0}, \lambda\}$ for every $j \in N \setminus \{i\}$. From the definition of the rule ϕ^{**}, we have: $\phi^{**}_j((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) = \phi^{**}_j(\mathbf{c}, E, \mathbf{p})$ for every $j \in N \setminus \{i\}$. That is, the rule ϕ^{**} satisfies *weak consistency.*

Observe, $c_{is_0} \geq \min_{s \in S} c_{is}$. Hence, $\min_{s \in S} c_{is} \geq E$ implies $c_{is_0} \geq E$. *No reward for more irrelevant claims* follows now by the definition of ϕ^{**}. Now, ϕ^{**} satisfies *no sudden response to uncertainty* vacuously. Also, for any "generic problem" $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, consider the corresponding profile $\mathbf{c}'$ in which, for a particular $i \in N$, $\mathbf{c}'_i := \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S$ and $\mathbf{c}'_j := \mathbf{c}_j$ for every $j \in N \setminus \{i\}$. If $\bar{c}_i(\mathbf{p}) \leq E$ and $c_{is_0} < \bar{c}_i(\mathbf{p})$, then $\phi^{**}_i(\mathbf{c}, E, \mathbf{p}) < \phi^{**}_i(\mathbf{c}', E, \mathbf{p})$. Since $\bar{c}_i(\mathbf{p}) = \bar{c}'_i(\mathbf{p})$ and $\mathbf{c}'_j = \mathbf{c}_j$ for every $j \in N \setminus \{i\}$, this constitutes a violation of *no penalty for risk.* The presentation of this example is complete. ■

11. Conclusion

In a framework of state contingent claims, where the resource needs to be divided before realization of state of the nature, we introduce and characterize two wide classes of rules, namely, ex–ante and ex–post rules. To characterize these classes of rules we use adaptations of standard axioms from the literature as well as some natural axioms on uncertainty and risk. With a finite set of states for representation of contingent claims, we model a very general setting that allows heterogeneity and correlation among individual beliefs.

We first introduce and characterize the ex–ante forms of a class of parametric rules. The salient feature of these rules is invariance to more irrelevant claims which has a huge normative appeal. This axiom leaves us with a wide range of a very rich class of rules. Some notable rules in this class include ex–ante forms of Talmud rules and priority augmented constrained equal awards rules. Next, we establish that the ex–post forms of rationing rules are equivalent to a condition satisfied by the preferences of risk neutral individual which we term indifference to independent combinations of gambles. This axiom, relating to a risk neutral individual's preference, establishes the precise normative characteristic of any ex–post rule. Finally, we provide a partial comparative static where it is shown that individuals will prefer an ex–ante rule to ex–post rule, whenever the level of the resource to be divided is sufficiently low.

We identify two interesting directions for future research. Although, the No Reward for More Irrelevant Claims is a compelling axiom, it does exclude some of the rules; *e.g.*, proportional rules. It would be interesting to venture into the possibility of obtaining a slightly more general result which would include such rules within this framework. Finally, it would be a natural excercise to deploy the general characterizations presented in this article in combination with standard axiomatic analysis of the classical theory to pin down specific families of rules with the ex–ante and the ex–post forms.

12. Appendix — Proof of results

Proof of Theorem 1. Observe, it is enough to argue, for any $h \in \mathscr{H}$ and any $T \in \mathscr{T}$, $\phi^{h,T}$ is indeed a rule and satisfies *weak consistency*, *claim monotonicity* and *no reward for more irrelevant claims*. We proceed to establish each of these four objectives in turn.

Claim 1. $\phi^{h,T}$ is a *rule* and satisfies *no reward for more irrelevant claims*: Fix any problem $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. By condition $R.1$, we have $(T(\mathbf{c}, \mathbf{p}), E) \in \mathscr{D}^*$ as $\sum_{i \in N} c_{is} \geq E$ for every $s \in S$. Define $t := E$. For every $i \in N$, denote by x_i^t the "truncated claim"; *i.e.*, $x_i^t := \min\{T_i(\mathbf{c}, \mathbf{p}), t\}$. By conditions $H.1$ and $H.2$, it follows that there exists a unique $\theta \in [\theta_*, \theta^*]$ that solves the equation, in θ, $\sum_{i \in N} h_i(\theta, x_i^t) = t$. Thus, the map $\phi^{h,T}$ in indeed a rule. Further, it is clear that the rule satisfies *no reward for more irrelevant claims* by condition $R.2$. To see this, note that $T_i(\mathbf{c}, \mathbf{p}) \geq E$ if $\min_{s \in S} c_{is} \geq E$. This completes the proof of claim 1.

Claim 2. $\phi^{h,T}$ satisfies *weak consistency*: Fix any problem $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. By condition $R.1$, we have $(T(\mathbf{c},\mathbf{p}), E) \in \mathscr{D}^*$ as $\sum_{i\in N} c_{is} \geq E$ for every $s \in S$. Define $t := E$. For every $i \in N$, denote by x_i the "truncated claim"; *i.e.*, $x_i := \min\{T_i(\mathbf{c},\mathbf{p}), t\}$. For $\mathbf{c}'_i \geq \mathbf{c}_i$, let $x'_i := \min\{T_i((\mathbf{c}'_i, \mathbf{c}_{-i}), \mathbf{p}), t\}$. Let θ and θ' satisfy $\sum_{j\in N} h_j(\theta, x_j) = t$ and $h_i(\theta', x'_i) + \sum_{j\in N\setminus\{i\}} h_j(\theta, x_j) = t$, respectively. We consider $\mathbf{c}'_i$ such that $h_i(\theta', x'_i) = h_i(\theta, x_i)$. We must argue: $h_j(\theta', x_j) = h_j(\theta, x_j)$ for every $j \in N \setminus \{i\}$.

If $\theta = \theta'$, then we have nothing to show. Hence, we consider $\theta \neq \theta'$. Assume, without loss of generality, $\theta' > \theta$. By condition $H.3$, $h_j(\theta', x_j) > h_j(\theta, x_j)$ for every $j \in N \setminus \{i\}$. Hence, $\sum_{j\in N\setminus\{i\}} h_j(\theta', x_j) > \sum_{j\in N\setminus\{i\}} h_j(\theta, x_j)$. Since $h_i(\theta', x'_i) = h_i(\theta, x_i)$, we obtain:

$$h_i(\theta', x'_i) + \sum_{j\in N\setminus\{i\}} h_j(\theta', x_j) > \sum_{j\in N} h_j(\theta, x_j).$$

However, both sides of the inequality equal t by definition of θ' and θ, respectively. Thus, we have a contradiction. Hence, our supposition that $\theta' > \theta$ is wrong. This completes the proof of claim 2.

Claim 3. $\phi^{h,T}$ satisfies *claim monotonicity*: Fix any problem $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. By condition $R.1$, we have $(T(\mathbf{c},\mathbf{p}), E) \in \mathscr{D}^*$ as $\sum_{i\in N} c_{is} \geq E$ for every $s \in S$. Define $t := E$. For every $i \in N$, denote by x_i the "truncated claim"; *i.e.*, $x_i := \min\{T_i(\mathbf{c},\mathbf{p}), t\}$. Also, let $\mathbf{c}'_i \geq \mathbf{c}_i$, and let $x'_i := \min\{T_i((\mathbf{c}'_i, \mathbf{c}_{-i}), \mathbf{p}), t\}$. Observe, by condition $R.3$, $T_i((\mathbf{c}'_i, \mathbf{c}_{-i}), \mathbf{p}) \geq T_i(\mathbf{c},\mathbf{p})$. Thus, $x'_i \geq x_i$. Let θ and θ' satisfy $\sum_{j\in N} h_j(\theta, x_j) = t$ and $h_i(\theta', x'_i) + \sum_{j\in N\setminus\{i\}} h_j(\theta, x_j) = t$, respectively. We must argue: $h_i(\theta', x'_i) \geq h_i(\theta, x_i)$.

If $\theta = \theta'$, then we have nothing to show. Hence, we consider $\theta \neq \theta'$. Suppose, $\theta' > \theta$. By condition $H.3$, $h_j(\theta', x_j) > h_j(\theta, x_j)$ for every $j \in N$. As $x'_i \geq x_i$, it follows from condition $H.4$ that $h_i(\theta', x'_i) \geq h_i(\theta', x_i)$. Hence, we obtain $h_i(\theta', x'_i) \geq h_i(\theta, x_i)$. Thus, we have: $h_i(\theta', x'_i) + \sum_{j\in N\setminus\{i\}} h_j(\theta', x_j) > \sum_{j\in N} h_j(\theta, x_j)$. However, both sides of the inequality equal t by definition of θ' and θ, respectively. Thus, we have a contradiction. Hence, our supposition that $\theta' > \theta$ is wrong. Therefore, $\theta' < \theta$. Then $h_j(\theta', x_j) < h_j(\theta, x_j)$ for every $j \in N \setminus \{i\}$. It follows: $\sum_{j\in N\setminus\{i\}} h_j(\theta', x_j) < \sum_{j\in N\setminus\{i\}} h_j(\theta, x_j)$. Together with the definition of θ and θ', we conclude that $x'_i > x_i$ as required. This completes the proof of claim 3. ■

Proof of Lemma 1. The proof entails the construction of a list of $|N|$ new profile of claims, recursively by starting with a given problem, such that the analogues of implications 1 and 2 are preserved across any two successive profiles. Then, implications 1 and 2 of the lemma shall be claimed to be established by showing that the $|N|$th. element of the list is indeed equal to $\mathbf{c}^*(\mathbf{c},\mathbf{p})$.

The argument is organized via several steps. Fix any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ such that $E \leq \min_{j\in N} \bar{c}_j(\mathbf{p})$.

Step 1: We construct a list of profiles $\langle \mathbf{c}^k : k \in \{0, 1, \ldots, |N|\}\rangle$ as follows:

$C.1$ For $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, define $\mathbf{c}^0 := \mathbf{c}$.

$C.2$ Suppose, for some $k \in \{1, \ldots, |N|\}$, $\mathbf{c}^{k-1}$ has been defined. Then $\mathbf{c}^k$ is obtained as follows: define $\mathbf{c}^k := \langle \mathbf{c}_i^k \in \mathbb{R}_+^S : i \in N\rangle$, where, for any $i \in N$, $\mathbf{c}_i^k$ is defined by:

$$\mathbf{c}_i^k := \begin{cases} \bar{c}_k^{k-1}(\mathbf{p}) \cdot \mathbf{1}_S & ; \text{if } i = k; \\ \mathbf{c}_i^{k-1} & ; \text{if } i \neq k. \end{cases}$$

This completes the construction.

Step 2: We shall now show, for any $k \in \{1, 2, \ldots, |N|\}$, the truth of the statement $St[k]$ which is the conjunction of the following statements:

$k.1$ $(\mathbf{c}^k, E, \mathbf{p}) \in \mathscr{D}$.

$k.2$ $\phi(\mathbf{c}^k, E, \mathbf{p}) = \phi(\mathbf{c}^{k-1}, E, \mathbf{p})$.

$k.3$ $\mathbf{c}_k^k = \bar{c}_k^{k-1}(\mathbf{p}) \cdot \mathbf{1}_S$.

$k.4$ *For any* $k \in \{1, 2, \ldots, |N|\}$ *and any* $i \in N \setminus \{k\}$, $\mathbf{c}_i^k = \mathbf{c}_i^{k-1}$.

$k.5$ $\bar{c}_k^k(\mathbf{p}) \geq E$.

First, we prove $St[1]$; *i.e.*, $k = 1$. Observe, 1.3 and 1.4 are trivial by the definition of $\mathbf{c}_i^1$ in $C.2$. To establish 1.1, we must argue that $\sum_{i\in N} c_{is}^1 \geq E$ for every $s \in S$. So, fix any $s \in S$. By $C.1$, we obtain $\mathbf{c}_1^0 = \mathbf{c}_1$. Thus, $\bar{c}_1^0(\mathbf{p}) = \bar{c}_1(\mathbf{p})$. By $C.2$, we have $c_{1s}^1 = \bar{c}_1(\mathbf{p})$. Since $\min_{i\in N} \bar{c}_i(\mathbf{p}) \geq E$, it follows $c_{1s}^1 \geq E$ which also proves 1.5. Hence, $\sum_{i\in N} c_{is}^1 \geq c_{1s}^1 \geq E$. Thus, 1.1 stands proven. With 1.1 established, both $\phi(\mathbf{c}^1, E, \mathbf{p})$ and $\phi(\mathbf{c}^0, E, \mathbf{p})$ are well–defined. We next argue that 1.2 holds; *i.e.*, the two allocations are equal. For this, it will be enough to argue that $\phi_1(\mathbf{c}^1, E, \mathbf{p}) = \phi_1(\mathbf{c}^0, E, \mathbf{p})$; for then $\sum_{i\neq 1} \phi_i(\mathbf{c}^1, E, \mathbf{p}) = \sum_{i\neq 1} \phi_i(\mathbf{c}^0, E, \mathbf{p})$ which, by weak consistency, implies $\phi_i(\mathbf{c}^1, E, \mathbf{p}) = \phi_i(\mathbf{c}^0, E, \mathbf{p})$ for every $i \neq 1$. To argue that $\phi_1(\mathbf{c}^1, E, \mathbf{p}) = \phi_1(\mathbf{c}^0, E, \mathbf{p})$, we begin by defining $\mathbf{c}_1' := \langle c_{1s}' \in \mathbb{R}_+ : s \in S\rangle$ by $c_{1s}' := \max\{c_{1s}^0, \bar{c}_1^0(\mathbf{p})\}$ for every $s \in S$. Thus, $\mathbf{c}_1' \geq \mathbf{c}_1^0$. By claim monotonicity, we have $\phi_1((\mathbf{c}_1', \mathbf{c}_{-1}^0), E, \mathbf{p}) \geq \phi_1(\mathbf{c}^1, E, \mathbf{p})$. Since $E \leq \min_{j\in N} \bar{c}_j(\mathbf{p})$ and $\mathbf{c}^0 = \mathbf{c}$, we have $\bar{c}_1^0(\mathbf{p}) \geq E$. Hence, $\min_{s\in S} c_{1s}' \geq E$. By no reward for more irrelevant claims, we have

$\phi_1((\mathbf{c}'_1, \mathbf{c}^0_{-1}), E, \mathbf{p}) \leq \phi_1(\mathbf{c}^0, E, \mathbf{p})$. Thus, $\phi_1(\mathbf{c}^1, E, \mathbf{p}) \leq \phi_1(\mathbf{c}^0, E, \mathbf{p})$. Since $\mathbf{c}^1_1 = \bar{c}^0_1(\mathbf{p}) \cdot \mathbf{1}_S$, by no penalty for risk it follows that $\phi_1(\mathbf{c}^1, E, \mathbf{p}) \geq \phi_1(\mathbf{c}^0, E, \mathbf{p})$. Hence, $\phi_1(\mathbf{c}^1, E, \mathbf{p}) = \phi_1(\mathbf{c}^0, E, \mathbf{p})$. This completes the proof of $St[1]$.

Now, for any $k \in \{2, 3, \ldots, |N|\}$, we prove $St[k]$ assuming the truth of $St[l]$ for every $l \in \{1, 2, \ldots, k-1\}$. Observe, $k.3$ and $k.4$ are trivial by the definition of $\mathbf{c}^k_i$ in $C.2$. To establish $k.1$, we must argue that $\sum_{i\in N} c^k_{is} \geq E$ for every $s \in S$. So, fix any $s \in S$. By the conjunction of $l.4$ of $St[l]$ for every $l \in \{1, 2, \ldots, k-1\}$, we have $\mathbf{c}^{k-1}_k = \mathbf{c}^0_k$. Together with $C.1$, we obtain $\mathbf{c}^{k-1}_k = \mathbf{c}_k$. Thus, $\bar{c}^{k-1}_k(\mathbf{p}) = \bar{c}_k(\mathbf{p})$. It follows from $k.3$ that $c^k_{ks} = \bar{c}_k(\mathbf{p})$. Since $\min_{i\in N} \bar{c}_i(\mathbf{p}) \geq E$, it follows $c^k_{ks} \geq E$ which also proves $k.5$. Hence, $\sum_{i\in N} c^k_{is} \geq c^k_{ks} \geq E$. Thus, $k.1$ stands proven. With $k.1$ established, and the truth of $St[k-1]$ assumed, both $\phi(\mathbf{c}^k, E, \mathbf{p})$ and $\phi(\mathbf{c}^{k-1}, E, \mathbf{p})$ are well–defined. We next argue that $k.2$ holds; *i.e.*, the two allocations are equal. For this, it will be enough to argue that $\phi_k(\mathbf{c}^k, E, \mathbf{p}) = \phi_k(\mathbf{c}^{k-1}, E, \mathbf{p})$; for then $\sum_{i\neq k} \phi_i(\mathbf{c}^k, E, \mathbf{p}) = \sum_{i\neq k} \phi_i(\mathbf{c}^{k-1}, E, \mathbf{p})$ which, by weak consistency, implies $\phi_i(\mathbf{c}^k, E, \mathbf{p}) = \phi_i(\mathbf{c}^{k-1}, E, \mathbf{p})$ for every $i \neq k$. To argue that $\phi_k(\mathbf{c}^k, E, \mathbf{p}) = \phi_k(\mathbf{c}^{k-1}, E, \mathbf{p})$, we begin by defining $\mathbf{c}'_k := \langle c'_{ks} \in \mathbb{R}_+ : s \in S\rangle$ by $c'_{ks} := \max\{c^{k-1}_{ks}, \bar{c}^{k-1}_k(\mathbf{p})\}$ for every $s \in S$. Thus, $\mathbf{c}'_k \geq \mathbf{c}^{k-1}_k$. By claim monotonicity, we have $\phi_k((\mathbf{c}'_k, \mathbf{c}^{k-1}_{-k}), E, \mathbf{p}) \geq \phi_k(\mathbf{c}^k, E, \mathbf{p})$. By $(k-1).5$, $\bar{c}^{k-1}_k(\mathbf{p}) \geq E$. Hence, $\min_{s\in S} c'_{ks} \geq E$. By no reward for more irrelevant claims, we have $\phi_k((\mathbf{c}'_k, \mathbf{c}^{k-1}_{-k}), E, \mathbf{p}) \leq \phi_k(\mathbf{c}^{k-1}, E, \mathbf{p})$. Thus, $\phi_k(\mathbf{c}^k, E, \mathbf{p}) \leq \phi_k(\mathbf{c}^{k-1}, E, \mathbf{p})$. Since $\mathbf{c}^k_k = \bar{c}^{k-1}_k(\mathbf{p}) \cdot \mathbf{1}_S$, by no penalty for risk it follows that $\phi_k(\mathbf{c}^k, E, \mathbf{p}) \geq \phi_k(\mathbf{c}^{k-1}, E, \mathbf{p})$. Hence, $\phi_k(\mathbf{c}^k, E, \mathbf{p}) = \phi_k(\mathbf{c}^{k-1}, E, \mathbf{p})$. This completes the proof of $St[k]$ assuming the truth of $St[l]$ for every $l \in \{1, 2, \ldots, k-1\}$. As $St[1]$ has already been shown to be true above, we conclude that $St[k]$ holds for every $k \in \{1, 2, \ldots, |N|\}$.

Step 3: We shall argue that $\mathbf{c}^{|N|} = \mathbf{c}^*(\mathbf{c}, \mathbf{p})$; *i.e.*, we must argue $\mathbf{c}^{|N|}_i = \mathbf{c}^*_i(\mathbf{c}, \mathbf{p})$ for any $i \in N$. Fix any $i \in N$. By definition, $\mathbf{c}^*_i(\mathbf{c}, \mathbf{p}) = \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S$. Hence, we must argue that $\mathbf{c}^{|N|}_i = \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S$. Let $k := i$. By the conjunction of the statements $St[1]$ to $St[k-1]$, we have $\mathbf{c}^{k-1}_k = \mathbf{c}^0_k$. Also, $\mathbf{c}^0_k = \mathbf{c}_k$ by $C.1$. Thus, $\mathbf{c}^{k-1}_k = \mathbf{c}_k$; hence, $\bar{c}^{k-1}_k(\mathbf{p}) = \bar{c}_k(\mathbf{p})$. By $k.3$ of $St[k]$, we obtain $\mathbf{c}^k_k = \bar{c}_k(\mathbf{p}) \cdot \mathbf{1}_S$. If $k = |N|$, then $\mathbf{c}^{|N|}_k = \mathbf{c}^k_k$. However, if $k \in \{1, 2, \ldots, |N|-1\}$, then by the conjunction of the statements $St[k+1]$ to $St[|N|]$ we have $\mathbf{c}^{|N|}_k = \mathbf{c}^k_k$. Thus, $\mathbf{c}^{|N|}_k = \bar{c}_k(\mathbf{p}) \cdot \mathbf{1}_S$. Since k was defined to be i, we have established that $\mathbf{c}^{|N|}_i = \mathbf{c}^*_i(\mathbf{c}, \mathbf{p})$. As $i \in N$ was arbitrary, we have $\mathbf{c}^{|N|} = \mathbf{c}^*(\mathbf{c}, \mathbf{p})$.

Step 4: We conclude by showing that implications 1 and 2 of the lemma indeed hold. Since $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ and $\mathbf{c}^0 = \mathbf{c}$ by $C.1$, via the conjunction of $k.1$ of $St[k]$ for every $k \in \{1, 2, \ldots, |N|\}$ we have $(\mathbf{c}^{|N|}, E, \mathbf{p}) \in \mathscr{D}$. Now, by an argument obtained by replacing "$k.1$" with "$k.2$" in the previous sentence, we obtain $\phi(\mathbf{c}^{|N|}, E, \mathbf{p}) = \phi(\mathbf{c}, E, \mathbf{p})$. Finally, by the previous step, we conclude that $(\mathbf{c}^*(\mathbf{c}, \mathbf{p}), E, \mathbf{p}) \in \mathscr{D}$ and $\phi(\mathbf{c}^*(\mathbf{c}, \mathbf{p}), E, \mathbf{p}) = \phi(\mathbf{c}, E, \mathbf{p})$. This completes the proof. ■

Proof of Theorem 2. The "if" part is obvious. We proceed to establish the "only if" part. First, we define a map $\psi : \mathscr{D}^* \to \mathbb{R}^N_+$. For any $(\mathbf{x}, t) \in \mathscr{D}^*$ with $\mathbf{x} \equiv \langle x_i \in \mathbb{R}_+ : i \in N \rangle$, let $E := t$ and $\mathbf{c} := \langle \mathbf{c}_i \in \mathbb{R}^S_+ : i \in N \rangle$, where $\mathbf{c}_i := x_i \cdot \mathbf{1}_S$ for every $i \in N$. Observe, $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. Then set $\psi(\mathbf{x}, t) := \phi(\mathbf{c}, E, \mathbf{p})$. Since ϕ satifies $\sum_{i \in N} \phi_i(\mathbf{c}, E, \mathbf{p}) = E$, we obtain $\sum_{i \in N} \psi_i(\mathbf{x}, t) = t$. To complete the proof, it is enough to show that $\phi(\mathbf{c}, E, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E)$ for any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$, where $\bar{c}(\mathbf{p}) \equiv \langle \bar{c}_i(\mathbf{p}) : i \in N \rangle$. Fix any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$. By definition of $\mathscr{D}$, it follows that the corresponding $(\bar{c}(\mathbf{p}), E) \in \mathscr{D}^*$. If $E \leq \min_{i \in N} \bar{c}_i(\mathbf{p})$, then by Lemma 1 we obtain $\phi(\mathbf{c}, E, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E)$. Observe, $\mathbf{c}^*(\mathbf{c}, \mathbf{p}) = \bar{c}(\mathbf{p})$. Thus, for any $E \leq \min_{i \in N} \bar{c}_i(\mathbf{p})$, we have established $\phi(\mathbf{c}, E, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E)$.

Suppose, for the sake of contradiction, the set $\mathscr{S} := \{E \in \mathscr{E} : \phi(\mathbf{c}, E, \mathbf{p}) \neq \psi(\bar{c}(\mathbf{p}), E)\}$ is not empty. Since $\mathscr{S}$ is bounded below by $\min_{i \in N} \bar{c}_i(\mathbf{p})$, $E_* := \inf \mathscr{S}$ exists and $E_* \geq \min_{i \in N} \bar{c}_i(\mathbf{p})$. Without loss of any generality, by continuity of ϕ, we shall assume $\mathbf{p} \in \Delta^o(S)$. If $\min_{i \in N} \bar{c}_i(\mathbf{p}) = 0$, then $\phi_j(\mathbf{c}, E, \mathbf{p}) = 0$ for any $j \in \arg\min_{i \in N} \bar{c}_i(\mathbf{p})$. Thus, we may assume without any loss of generality that $\min_{i \in N} \bar{c}_i(\mathbf{p}) > 0$, Hence, E_* is strictly positive. From the definition of E_*, for any $E < E_*$, $\phi(\mathbf{c}, E, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E)$. Now, ψ being defined via ϕ, is also continuous. Thus, $\phi(\mathbf{c}, E_*, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E_*)$. By no sudden response to uncertainty, there exists $\epsilon > 0$ such that, for any $E \in [E_*, E_* + \epsilon)$, it must be that $\phi(\mathbf{c}, E, \mathbf{p}) = \psi(\bar{c}(\mathbf{p}), E)$. This violates the definition of E_* as the infimum of the set $\mathscr{S}$. Thus, we have a contradiction to the supposition that $\mathscr{S}$ is not empty. This completes the proof. ■

Proof of Theorem 3. Consider an individual i who is driven by von Neumann–Morgenstern preferences over money lotteries. Since the class of all Bernoullians whose expected utility represents such a preference is invariant under postitive affine transformations, we consider a Bernoullian $u_i : \mathbb{R}_+ \to \mathbb{R}$ such that $u_i(0) = 0$ and $u_i(1) = 1$. Let the individual be risk neutral. Thus, $u_i(x) = x$ for every $x \in \mathbb{R}_+$. Recall, the definition of the map $U : \mathscr{I}_{\mathscr{D}} \to \mathbb{R}$. For any $\bigotimes_{m=1}^{M} [\bigoplus_{k=1}^{K_m} \pi^m_k \bullet (\mathbf{v}^m_k, \mathbf{p}^m_k)]$ element of $\mathscr{I}_{\mathscr{D}}$:

$$U\left(\bigotimes_{m=1}^{M} [\bigoplus_{k=1}^{K_m} \pi^m_k \bullet (\mathbf{v}^m_k, \mathbf{p}^m_k)]\right) := \sum_{\langle k_m \leq K_m : m \leq M \rangle} [\prod_{m=1}^{M} \pi^m_{k_m}] \cdot u_i\Big(\sum_{m=1}^{M} \phi_i(\mathbf{v}^m_{k_m}, \mathbf{p}^m_{k_m})\Big).$$

Since $u_i(x) = x$ for every $x \in \mathbb{R}_+$ and $\sum_{k=1}^{K_m} \pi_k = 1$ for every $m \in \{1, 2, \ldots, M\}$, it follows that:

$$\sum_{\langle k_m \leq K_m : m \leq M \rangle} [\prod_{m=1}^{M} \pi_{k_m}^m] \cdot u_i\Big(\sum_{m=1}^{M} \phi_i(\mathbf{v}_{k_m}^m, \mathbf{p}_{k_m}^m)\Big) = \sum_{m=1}^{M} \sum_{k=1}^{K_m} \pi_k^m \cdot \phi_i(\mathbf{v}_k^m, \mathbf{p}_k^m).$$

Hence, $U(\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]) = \sum_{m=1}^{M} \sum_{k=1}^{K_m} \pi_k^m \cdot \phi_i(\mathbf{v}_k^m, \mathbf{p}_k^m)$. Consider any two elements $\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)]$ and $\bigotimes_{m=1}^{M'}[\bigoplus_{k=1}^{K'_m} \pi'^m_k \bullet (\mathbf{v}'^m_k, \mathbf{p}'^m_k)]$ in $\mathscr{I}_{\mathscr{D}}$. By definition of $\succsim_i$, $\bigotimes_{m=1}^{M}[\bigoplus_{k=1}^{K_m} \pi_k^m \bullet (\mathbf{v}_k^m, \mathbf{p}_k^m)] \succsim_i \bigotimes_{m=1}^{M'}[\bigoplus_{k=1}^{K'_m} \pi'^m_k \bullet (\mathbf{v}'^m_k, \mathbf{p}'^m_k)]$ is, therefore, equivalent to the following inequality:

$$\sum_{m=1}^{M} \sum_{k=1}^{K_m} \pi_k^m \cdot \phi_i(\mathbf{v}_k^m, \mathbf{p}_k^m) \geq \sum_{m=1}^{M'} \sum_{k=1}^{K'_m} \pi'^m_k \cdot \phi_i(\mathbf{v}'^m_k, \mathbf{p}'^m_k).$$

With this expression in place, for a risk neutral individual i, we proceed to establish the claim of the theorem.

First, we prove the "if" part. Let, for every $i \in N$, the induced preference $\succsim_i$ satisfy indifference to independent combinations. Fix any $(\mathbf{v}, \mathbf{p}) \in \mathscr{D}$. We must argue: $\phi_i(\mathbf{v}, \mathbf{p}) = \sum_{s \in S} p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$ for any $i \in N$. Fix any $i \in N$. Recall, for $\mathbf{v} \equiv (\mathbf{c}, E)$, the corresponding zero situation is $\mathbf{v}_0 \equiv (\mathbf{c}, 0)$. By the definition of a rule, we have $\phi_i(\mathbf{v}_0, \mathbf{q}) = 0$ for any $\mathbf{q} \in \Delta(S)$. Thus, for any $s \in S$, $p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s) + (1 - p_s) \cdot \phi_i(\mathbf{v}_0, \mathbf{q}) = p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$. Hence, $\sum_{s \in S}[p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s) + (1 - p_s) \cdot \phi_i(\mathbf{v}_0, \mathbf{q})] = \sum_{s \in S} p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$. Since $\succsim_i$ satisfies indifference to independent combinations, $(\mathbf{v}, \mathbf{p}) \sim_i \bigotimes_{s \in S}[p_s \bullet (\mathbf{v}, \boldsymbol{\delta}_s) \bigoplus (1 - p_s) \bullet (\mathbf{v}_0, \mathbf{q})]$. By the definition of $\succsim_i$, we obtain $\phi_i(\mathbf{v}, \mathbf{p}) = \sum_{s \in S} p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$. Since $i \in N$ is arbitrary, the proof of the "if" part is complete.

Second, we prove the "only if" part. Let ϕ satisfy: for any $(\mathbf{v}, \mathbf{p}) \in \mathscr{D}$ and any $i \in N$, $\phi_i(\mathbf{v}, \mathbf{p}) = \sum_{s \in S} p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$. Fix any $i \in N$. We must argue: $\bigotimes_{s \in S}[p_s \bullet (\mathbf{v}, \boldsymbol{\delta}_s) \bigoplus (1 - p_s) \bullet (\mathbf{v}_0, \mathbf{q})] \sim_i (\mathbf{v}, \mathbf{p})$ for any $(\mathbf{v}, \mathbf{p}) \in \mathscr{D}$ and $\mathbf{q} \in \Delta(S)$. Fix $(\mathbf{v}, \mathbf{p}) \in \mathscr{D}$ and $\mathbf{q} \in \Delta(S)$. By the definition of a rule, we have $\phi_i(\mathbf{v}_0, \mathbf{q}) = 0$ for any $\mathbf{q} \in \Delta(S)$. Thus, for any $s \in S$, $p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s) + (1 - p_s) \cdot \phi_i(\mathbf{v}_0, \mathbf{q}) = p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$. Hence, $\sum_{s \in S}[p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s) + (1 - p_s) \cdot \phi_i(\mathbf{v}_0, \mathbf{q})] = \sum_{s \in S} p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s)$. That is, $\sum_{s \in S}[p_s \cdot \phi_i(\mathbf{v}, \boldsymbol{\delta}_s) + (1 - p_s) \cdot \phi_i(\mathbf{v}_0, \mathbf{q})] = \phi_i(\mathbf{v}, \mathbf{p})$. By definition of $\succsim_i$, we obtain $\bigotimes_{s \in S}[p_s \bullet (\mathbf{v}, \boldsymbol{\delta}_s) \bigoplus (1 - p_s) \bullet (\mathbf{v}_0, \mathbf{q})] \sim_i (\mathbf{v}, \mathbf{p})$. This completes the proof of the "only if" part. ■

Proof of Theorem 4. Fix $h \in \mathscr{H}$ and $i \in N$. Let ϕ^{EA} and ϕ^{EP} be the ex–ante and the ex–post rules defined by h, respectively. Fix any $(\mathbf{c}, E, \mathbf{p}) \in \mathscr{D}$ such that $\mathbf{c}_j$ is deterministic for every $j \in N \setminus \{i\}$. Define $\mathbf{c}'_i := \bar{c}_i(\mathbf{p}) \cdot \mathbf{1}_S$. Since $\|(\mathbf{c}_i, \mathbf{p})\|_1 = \|(\mathbf{c}'_i, \mathbf{p})\|_1$, it follows that $\phi^{EA}(\mathbf{c}, E, \mathbf{p}) = \phi^{EA}((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$. Since the profile $(\mathbf{c}'_i, \mathbf{c}_{-i})$ consists of deterministic claims by every individual, we have $\phi^{EA}((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) = \phi^{EP}((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$. That is, $\phi^{EA}(\mathbf{c}, E, \mathbf{p}) = \phi^{EP}((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p})$. Since

$E \leq \min_{j\in N} \bar{c}_j(\mathbf{p})$, from the proof of Lemma 1, we conclude that $\phi^{EP}((\mathbf{c}'_i, \mathbf{c}_{-i}), E, \mathbf{p}) \geq \phi^{EP}(\mathbf{c}, E, \mathbf{p})$. Hence, we obtain $\phi^{EA}(\mathbf{c}, E, \mathbf{p}) \geq \phi^{EP}(\mathbf{c}, E, \mathbf{p})$. Since h is made of strictly increasing maps h_i for any individual i, it follows that the inequality is indeed strict whenever $E > \min_{s\in S} c_{is}$ holds. ■

References

[1] Aumann, R. & M. Maschler (1985) "Game theoretic analysis of a bankruptcy problem from the Talmud", *Journal of Economic Theory*, **36**, pp. 195—213.

[2] van den Brink, R., Funaki, Y., & G. van der Laan (2013) "Characterization of the reverse Talmud bankruptcy rule by exemption and exclusion properties", *European Journal of Operational Research*, **228(2)**, pp. 413—417.

[3] van den Brink, R., & J. D. Moreno-Ternero (2017) "The reverse TAL-family of rules for bankruptcy problems", *Annals of Operations Research*, **254(1-2)**, pp. 449—465.

[4] Chun, Y. (1988) "The proportional solution for rights problems", *Mathematical Social Sciences*, **15**, pp. 231—246.

[5] Dagan, N. (1996) "New characterizations of old bankruptcy rules", *Social Choice and Welfare*, **13**, pp. 51—59.

[6] Ertemel S. & R. Kumar (2018) "Proportional rules for state contingent claims", *International Journal of Game Theory*, **47**, pp. 229—246.

[7] Flores-Szwagrzak, K. (2015) "Priority classes and weighted constrained equal awards rules for the claims problem", *Journal of Economic Theory*, **160**, pp. 36—55.

[8] Habis, H. & J–J. Herings (2011) "Transferable utility games with uncertainty", *Journal of Economic Theory*, **146**, pp. 973—988.

[9] Habis, H. & J–J. Herings (2013) "Stochastic bankruptcy games", *International Journal of Game Theory*, **42**, pp. 2126—2139.

[10] Herrero, C. & A. Villar (2001) "The three musketeers: four classical solutions to bankruptcy problems", *Mathematical Social Sciences*, **39**, pp. 307—328.

[11] Hougaard, J. (2009) "An Introduction to Allocation Rules", *Springer Science & Business Media.*

[12] Hougaard, J. & Moulin, H. (2018) "Sharing the cost of risky projects", *Economic Theory*, **65**, pp. 663—679.

[13] Ju, B–G., Miyagawa, E. & T. Sakai (2007) "Non–Manipulable division rules in claim problems and generalizations", *Journal of Economic Theory*, **132**, pp. 1—26.

[14] Long, Y., Sethuraman, J. & J. Xue (2019) "Equal–quantiles rule in rationing with claim uncertainty", Research Collection School Of Economics. Available at: https://ink.library.smu.edu.sg/soe_research/2274/.

[15] Moulin, H. (1985) "Egalitarianism and utilitarianism in quasi–linear bargaining", *Econometrica*, **53**, pp. 49—67.

[16] Moulin, H. (1987) "Equal or proportional division of a surplus and other methods", *International Journal of Game Theory*, **16**, pp. 161—186.

[17] Moulin, H. (2002) "Axiomatic Cost and Surplus–Sharing", Chapter–6 of K. Arrow, A. Sen & K. Suzumura (Eds.) *The Handbook of Social Choice and Welfare (Vol. 1)* Netherlands: North–Holland.

[18] O'Neill, B. (1982) "A problem of rights arbitration from the Talmud", *Mathematical Social Sciences*, **2**, pp. 345—371.

[19] Roth, A. E. (1977) "The Shapley value as a von Neumann-Morgenstern utility", *Econometrica: Journal of the Econometric Society*, pp. 657—664.

[20] Thomson, W. (2003) "Axiomatic and game–theoretic analysis of bankruptcy and taxation problems: a survey", *Mathematical Social Sciences*, **45**, pp. 249—297.

[21] Thomson, W. (2015) "Axiomatic and game–theoretic analysis of bankruptcy and taxation problems: an update", *Mathematical Social Sciences*, **74**, pp. 41—59.

[22] Thomson, W. (2019) "How To Divide When There Isn't Enough From Aristotle, the Talmud, and Maimonides to the axiomatics of resource allocation", unpublished manuscript.

[23] Xue, J. (2018) "Fair division with uncertain needs", *Social Choice and Welfare*, **51**, pp. 105—136.

[24] Yeh, C. H. (2008) "Secured lower bound, composition up, and minimal rights first for bankruptcy problems", *Journal of Mathematical Economics*, **44**, pp. 925—932.

[25] Young, P. (1987*a*) "On dividing an amount according to individual claims or liabilities", *Mathematics of Operations Research*, **12**, pp. 398—414.

[26] Young, P. (1987*b*) "Progressive taxation and the equal sacrifice principle", *Journal of Public Economics*, **32**, pp. 203—214.

[27] Young, P. (1988) "Distributive justice in taxation", *Journal of Economic Theory*, **44**, pp. 321—335.

[28] Young, P. (1990) "Progressive taxation and equal sacrifice", *The American Economic Review*, **80**, pp. 253—266.