

A Game-Theoretical Approach to Islamic Law Disputes

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Abstract

Purpose: In this study, we aim to analyze notable distribution dispute cases from Islamic law history. We will assess these alongside resolutions proposed by historical authorities, some of which evolved into established Islamic case law. Additionally, we intend to apply classic fair division rules to these cases, providing alternative solutions. Using a game-theoretical approach, we plan to compare Islamic solutions with traditional division rules through axiomatic analysis. Our goal is to systematically explore the unique principles underpinning Islamic distributions.

Design/methodology/approach: In this study, we collate Islamic inheritance law disputes involving conflicting claims, unresolvable by primary Islamic law sources, from historical and modern texts. We formally model these as claims problems, surplus-sharing problems, and adapted claims problems. Concurrently, we gather the proposed solutions and historical backgrounds offered by the era's authorities and jurists. These solutions are axiomatically generalized into rules, while the axioms characterizing distribution rules are checked if they are aligned with Islamic norms and values. This approach facilitates a comparison between Islamic distributions and classic division rules.

Findings: The 'Awl and Radd doctrines, utilized in Islamic inheritance law, are axiomatically equivalent to the Proportional Rule, a prevalent non-Jewish division rule. These doctrines present solutions impervious to manipulation by legal heirs through rights transfer, unlike other possible distributions. Ibn 'Abbas' solution for Awliyya cases utilizes sequential priorities and diverges uniquely from classic fair division rules in the literature. Additionally, it is established that Abu Yusuf's (b. 729) distribution for a legal dispute is axiomatically identical to Abraham ibn Ezra's (b. 1089) division rule.

Research limitations/implications: There is a scarcity of existing studies that thoroughly explore contentious disputes over resource claims in Islamic law. Moreover, these studies frequently fail to provide a comprehensive analysis of different cases, raising concerns about their reliability. As such, rigorous attention is required for case collection prior to analysis. Future research could focus on the assembly of fair division problems from Islamic history and on the separate collection

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of Islamic sharing methods. This approach might lead to characterizations of various Islamic rules, thus highlighting unique Islamic principles. In future work, it is essential to conduct a comprehensive axiomatic examination of the cases and proposed solutions.

Practical implications: This study sheds light on knowledge gaps and paves the way for new research directions. It highlights the lack of attention given by the fair division literature to disputes in Islamic law, despite the abundance of contentious cases. The study highlights the utility of cooperative game theory for analyzing Islamic legal disputes. Accounting for unique Islamic norms and principles, it lays the groundwork for a nuanced understanding of legal dispute dynamics and outcomes. By employing an interdisciplinary approach, this study endeavors to bridge the divide between game theory and Islamic law.

Social implications: This study, while filling a crucial research gap, carries broader societal implications. By illuminating longstanding debates in Islamic law, it fosters a fresh understanding of legal dispute evolution and interpretation. Axiomatic differences in rulers' and jurists' methods offer invaluable insights within an Islamic context, enhancing comprehension of socio-cultural dynamics influencing legal decision-making. This research could shape legal discourse, inform policy, and stimulate scholarly, juristic, and societal dialogue, thereby cultivating a more holistic and enlightened approach to handling legal disputes in Islamic law.

Originality/value: This study is the first to examine Islamic law's historical legal disputes from a game-theoretical standpoint. Existing studies rarely collect distribution disputes systematically, and none scrutinize the axiomatic rationales underlying authorities' and jurists' distributions, opting instead to focus on historical backgrounds. While the fair division literature extensively examines disputes, it often overlooks those originating from Islamic law, which presents a rich source of disputes that can be modeled as fair division problems. This research makes a distinct contribution by incorporating disputes from Islamic law into the existing body of cooperative game theory literature.

Keywords: Islamic law, inheritance, fair division, claims problem, axiomatic characterization

1 Introduction

What is the most equitable way to allocate a resource when multiple parties are competing for it? This age-old question, inherent in human society, is an area of focus across a multitude of academic fields. One such field is microeconomics, particularly within the subfield of cooperative game theory. Researchers in this domain investigate the interplay between the total value of all competing claims on a resource and the actual value of the resource itself.

When the value of a resource is insufficient to satisfy all claims or demands, this situation is commonly identified as a *claims problem*. Conversely, when the value of a resource exceeds the total claims or demands, the scenario is referred to as a *surplus-sharing problem*.

Several instances that are modeled as fair division problems in the literature have roots extending back centuries. The Talmud, a foundational text in Judaism, provides the basis for many scenarios modeled as claims problems, which are extensively cited in academic discourse. The *Contested Garment Problem* from the Talmud serves as one such instance. In this case, two individuals claim ownership of a garment; one asserts a claim to the entire garment, while the other lays a claim to half (see Table 1). This dispute, framed as a claims problem, is resolved by allotting three-fourths and one-fourth to the claimants respectively, following a Jewish method known as the *Concede-and-Divide (CD) Rule* in literature. Alternatively, a non-Jewish method known as the *Proportional (P) Rule* would allocate two-thirds and one-third to the claimants respectively, as cited in (Epstein, 1935).

Table 1: Contested Garment Problem

	Claims	CD	P
Claimant 1	1	3/4	2/3
Claimant 2	1/2	1/4	1/3

In the literature, contemporary cost and surplus-sharing problems are frequently modeled and analyzed as fair division problems. For instance, Littlechild and Owen (1973) adopts a game-theoretical approach to model an aircraft landing problem. The author’s suggested solution is identical to the solution obtained through the application of Ibn Ezra’s method, which will be defined and elaborated upon in the subsequent sections. Pathak et al. (2021) studies the fair allocation of vaccines, ventilators, and antiviral treatments as a claims problem and Bergantiños and Moreno-Tertero (2020) models division of broadcasting revenue among various football clubs as a surplus-sharing problem.

Drawing parallels with the analyses of both the contested garment problem and the ventilator distribution problem during the COVID-19 pandemic, researchers systematically gather and scrutinize case studies involving distribution disputes from both ancient and contemporary times. These instances are subsequently cast as fair division problems. When practical solutions for these problems exist, researchers rigorously examine them to ascertain whether they can be generalized as sharing methods. The sharing methods are then evaluated axiomatically, given that axioms offer a distinctive characterization for each method, thereby facilitating a systematic comprehension of their underlying principles.

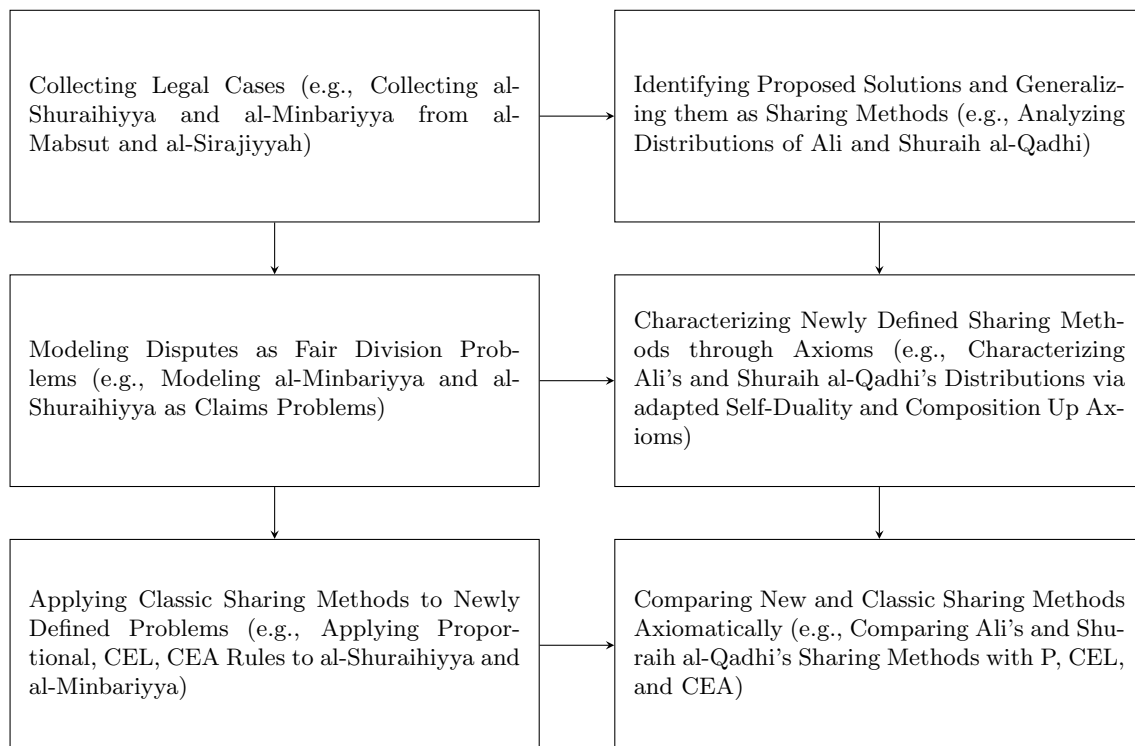


Figure 1

Furthermore, scholars apply traditional division rules from the literature to the problems collected, enabling a comparison between the solutions proposed by these rules and the newly formulated sharing methods. The methodology adopted within the literature and this study is illustrated in Figure 1.

While the history of Islam provides a multitude of legal cases that can be modeled as fair division problems and solutions that can be generalized as sharing methods, references to these cases are absent from the literature on fair division in microeconomics. This study endeavors to bridge this gap by scrutinizing Islamic legal cases involving distribution disputes, along with solutions proposed by rulers and jurists of that era, using reliable ancient and contemporary texts of Islamic law.

The three legal cases that are scrutinized and modeled as claims problems include the cases of al-Minbariyya, al-Shuraihiyya, and al-Mubahala. Solutions put forth by Ali, Shuraih al-Qadhi, ‘Umar, the ‘Awl doctrine, and Ibn ‘Abbas for these cases are also reviewed. Except for the solution proposed by Ibn ‘Abbas, the remaining solutions are generalized as sharing methods.

Additionally, two unnamed legal cases from Islamic history are examined and modeled as surplus-sharing problems. Solutions proposed by Ali, Zaid ibn Thabit, and the doctrine of Radd are documented. The distributions by Ali and the doctrine of Radd are identified as single-sharing methods, based on relevant axioms.

An anonymous case from the era of Abu Yusuf is analyzed and modeled as an adapted claims problem. Proposed solutions by Abu Yusuf and Muhammad al-Shaybani are recorded. The distribution proposed

by Abu Yusuf is generalized and recognized as a sharing method based on axioms.

The paper is organized as follows: Section 2 reviews the literature on fair division and the relevant Islamic law literature. Section 3 explains the fair division problems and the classic division rules from the literature, followed by the adaptation of claims problems, surplus-sharing problems, and adapted claims problems to Islamic law disputes. Finally, Section 4 offers concluding remarks to summarize the key findings and insights presented in the study.

2 Literature Review

In his pioneering study, O’Neill (1982) analyzes an ancient inheritance dispute from the Talmud and generalizes Ibn Ezra’s solution as Ibn Ezra’s Rule. The author also explores alternative sharing methods and proposes alternative resolutions to the analyzed disputes.

In the literature, different fair division rules are characterized by specific sets of axioms that serve to emphasize their normative justification. For instance, being the most widely known rule, the Proportional (P) Rule mentioned in Aristotle’s *The Nicomachean Ethics* and applied by Cicero in positive law (Engle, 2012), is characterized using axiomatic approach (De Frutos, 1999; Moulin, 1985; O’Neill, 1982). Studies have also examined legal disputes, including marriage contracts, conflicting claims, and bankruptcy cases, by modeling them as claims problems and characterizing proposed solutions as rules. Aumann and Maschler (1985) examine marriage contract disputes and bankruptcy cases from the Babylonian Talmud and characterize the Concede-and-Divide (CD) Rule, while Young (1987) models legal disputes outside the Talmud. Dagan (1996) model disputes from the Babylonian Talmud and characterizes the Constrained Equal Awards (CEA) Rule, and the Talmud (T) Rule. These studies also propose solutions based on fair division rules and apply them to various contexts. Moulin (1987) adapts fair division rules to the surplus-sharing problem and characterizes the Proportional (P) Rule and the Constrained Equal Losses (CEL) Rule. The relationship between resources and claims is also explored in cost-sharing problems (Moulin, 1987), taxation (Young, 1987), revenue sharing problems (Ginsburgh & Zang, 2001), fund distribution problems (Pulido et al., 2002), and so on.

In a more recent study, the resource allocation under risk and uncertainty is examined by Ertemel and Kumar (2018) using a proportional approach. Csóka and Herings (2021) investigate fair division rules in financial networks, while Heo and Lee (2023) models the problem of CO_2 emissions as the cost-sharing problem. For more about the literature, Moulin (2002) and Thomson (2019) provide comprehensive surveys on fair division.

In summary, the literature on fair division explores ancient and modern problems from real life with proposed solutions. The problems are formally defined as fair division problems and their solutions are generalized as sharing methods. These sharing methods are characterized by their properties, which are expressed as mathematical “axioms”, allowing for a better understanding of their properties and suitability for different contexts.

On the other hand, the concept of inheritance, as defined by the Romans, involves the transfer of an individual’s complete legal position upon their death (Du Plessis, 2015). Inheritance law encompasses



Figure 2: Formation of Legal Doctrine

a set of rules that govern the distribution of rights, debts, and other legal relationships of the deceased, known as the inheritor, which continue to be in effect after their passing. The placement of these rules on the spectrum of default-mandatory rules is a distinguishing factor among various inheritance law systems (Hirsch, 2004). Testamentary power, which grants the testator the authority to determine the allocation of their assets through a will, is also a significant characteristic of inheritance systems (Sitkoff & Dukeminier, 2017).

In contrast to the majority of legal systems, Islamic inheritance law diverges from relying on testamentary power and default-mandatory rules (Ghafar Ismail et al., 2014). Its primary objective is to ensure the material provision for the surviving family, relatives, and dependents who share mutual bonds and responsibilities with the deceased (Berkah & Sawarjuwono, 2019). Mandatory rules take precedence in the branch of inheritance in Islamic law, providing clear and rigid guidelines for asset distribution (Rohman, 2022). This branch, known as “ilm al-faraid” in Arabic, showcases the inflexibility of its rules (Anderson, 1965). Islamic law establishes a precise and exhaustive hierarchy of relatives, with firmly established entitlements. The term *legal heir* in the Islamic context exclusively refers to those relatives to whom property automatically passes by operation of law upon the owner’s demise. The rights of these legal heirs form the foundation of the entire succession system, as they are inherently irrevocable (Coulson, 1971).

According to Kelsen (1959-1960), every norm in a legal system is derived from the norm that is hierarchically superior to it. As Abdel-Wahab (1962) states that this hierarchy of legal norms in Islam is quite clear and uncontroversial. Schacht (1979) states that the classical theory of Islamic law categorizes the legal doctrine into four: the Qur’an, the Sunna, which is Prophet Mohammed’s model behavior, the ijma, which is the juridical consensus, and the qiyas, which is the method of analogy. The Qur’an is the basic norm that no legal rule applied should contradict. Then comes the Sunna, the divinely inspired behavior of Prophet Mohammed, which, together with the Qur’an, binds the judge to adopt a solution provided by one of these two sources (Abdel-Wahab, 1962). Together with the Qur’an and the Sunna, the secondary sources of Islam contribute to the formation of the legal doctrine (see Figure 2).

During his lifetime, Prophet Mohammed did not attempt to establish a comprehensive legal code, instead providing ad hoc solutions to emerging problems (Coulson, 1971). The traditions of early Islam (Sunna), which contain extensive commentary on various topics including contracts, taxes, property rights, and inheritance, do not cover all contemporary issues (Kuran, 1995). As a result, consultation with secondary Islamic sources becomes necessary when resolving specific disputes (Ali, 2016). This necessity can be easily observed in the famous legal disputes of Islamic inheritance law. Although legal heirs are clearly defined in the Qur’an, specifically in Surah al-Nisa’ (4:7, 8, 9, 11, 12, 33, 176), Surah al-Ahzab (33:6), as well as in the hadiths (Rasban et al., 2020), the actual inheritance law that is

applied by rulers and jurists seemingly is not solely based on the Qur'an and Sunna. (Souaiaia, 2005). For instance, if a woman dies leaving a husband, a mother, and a sister, the Qur'an and the Sunna command that the husband inherits half of the estate, the mother inherits one-third of the estate, and the sister inherits half of the estate (Souaiaia, 2005). Since the total value of these shares exceeds the value of the estate, the judgment cannot be rendered according to the Qur'an and Sunna only. In this case, Ali, the fourth caliph, decides that the heirs would bear the loss proportionally to their shares as cited in Ajetunmob (1988), Al-Sarakhsi (1087b), Coulson (1971), and Gandz (1938). On the other hand, Ibn 'Abbas, an early Qur'anic scholar, advises that the loss should be borne by the distant relative as cited in Jones and Rumsey (1890), Schacht (1979), Seydisehrī (1883), and Souaiaia (2005). These applications are distinguished by the approach of the legal doctrines to blood relations (Akhlaq, 2023; Cheema, 2012; Eich, 2012; Gupta, 2021; Hazleton, 2010; Holtmann, 2014; Shania, 2021). Sunni and Shia schools subsequently recognize these practical applications as rules. This type of inheritance problem, in which the total of the legal shares exceeds the estate and can therefore be modeled as a claims problem, is referred to as 'Awl in Islamic law history and has been extensively studied (Janin & Kahlmeyer, 2015). As Coulson (1971) states, in the literature, inheritance law disputes are examined according to the relationship between the net estate and the total legal shares. If the net estate's value is less than the total number of legal shares, the situation is known as Awliyya. If the value of the net estate exceeds the total number of legal shares, the situation is known as Raddiyya. If the value of the net estate is equal to the total number of legal shares, this is known as Adliyya.

Al-Khwarizmi was the first person to use algebraic operations to investigate Awliyya disputes, as cited by van der Waerden (2013). Al-Khwarizmi resolved problems in these disputes by citing the applicable laws. As cited by Gandz (1938), al-Khwarizmi referred to the operations he performed as inheritance algebra, not arithmetic. In a typical inheritance problem, Al-Khwarizmi first determined the legal heirs, then derived the pertinent laws, specified the method (e.g. the Doctrine of 'Awl) for solving the problem, and recommended the solution obtained by applying the method.

Al-Hidayah of Burhan al-Din al-Marghinani contains legal disputes with proposed resolutions by jurists of the time, pertaining to corporate law, law of obligations, and property law in the history of Islam. Particularly under the chapter entitled "Matter of Sharing," general distribution principles and legal disputes that can be modeled as cost-sharing problems are presented (Al-Marghinani, 1986).

Siraj al-Din al-Sajawandi explains which shares the deceased's relatives receive based on their kinship ties, along with the applicable laws, and then illustrates with concrete disputes in al-Sirajiyah. The author investigates legal disputes that qualify as Awliyya and Raddiyya cases and offers solutions supported by legal reasoning as cited in (Jones & Rumsey, 1890). Seydisehrī (1883) compiles and analyzes numerous legal disputes from the history of Islam in light of the applicable laws, including Awliyya cases that can be modeled as claims problems, and Raddiyya cases that can be modeled as surplus-sharing problems.

Coulson (1971) provides historical context for Islamic legal disputes and proposed solutions for them from various schools of thought in Islamic law, as well as legal justifications. The author asserts that, in general, Hanafi jurists apply proportionate reduction, also known as the doctrine of 'Awl, in Awliyya

cases and proportionate increase, known as the doctrine of Radd, in Raddiyya cases. According to the author, Shia jurists generally prioritize among the heirs rather than adjusting their shares. Naqvi (1972) appreciates the comprehensiveness of Coulson's take on the collection of Islamic legal disputes. Ajetunmob (1988) compiles legal disputes from the history of Islam that qualify as Awliyya and Adliyya cases. All of the disputes that the author investigates entail inheritance law. Despite the large number of collected cases, the author does not provide specific details about them, nor do they discuss the proposed solutions presented by authorities or the alternative solutions they provide. Powers (1993) and Bowen (1998) take socio-historical approach to justify the solutions proposed to Islamic inheritance disputes by secondary sources of Islam. The authors explore and analyze the social factors and implications surrounding de facto allocations provided by the authorities without explicitly offering detailed explanations or reasoning for them. Ahangar (2002) discusses the applicability of doctrine of 'Awl and Radd in Malaysian family law. Cilaro (2012) investigates the doctrine of 'Awl and Radd from a historical perspective. The author provides information regarding which schools of thought in Shia apply these principles.

In more recent studies, Yazıcı (2013) collects famous Awliyya cases from the history of Islam with proposed solutions by providing historical information for the cases and the solutions. Islam (2019) investigates the Islamic inheritance law in general and discusses the different doctrines that are applied in Sunni Law. By providing legal cases, the author explains the distributions with their historical context. However, no systematic analysis is made for the cases. Taqiyuddin (2021) investigates the justice provided by the application of 'Awl. The author finds that the procedure to resolve the Awliyya cases is the application of proportionate reduction, which is agreed upon by the majority of jurists. On the other hand, the author presents an alternative solution that opposes setting a priority among the heirs rather than adjusting their shares. The justifications for these methods are not made in accordance with the properties of these methods; instead, the author provides legal interpretations of Qur'anic verses. Arikewuyo (2023) examines the doctrine of 'Awl and Radd from the perspective of comparing literalism and rationalism. The author concludes that these applications serve as examples of the rationalist stream in Islam. However, the author does not provide a detailed discussion to support this analysis.

In summary, the literature on Islamic law that explores legal disputes which can be modeled as fair division problems is very limited in number. There is a lack of a systematic approach to handling these disputes in the available studies. The systematic studies mainly provide historical information and analysis on the social structure of societies where the rules are applied, as well as interpretations of rules that were applied or not. On the other hand, other studies focus on collecting cases and presenting solutions without further analysis or examination.

3 Methodology

Economics deploys a methodological agenda that simplifies the complexity of economic events by reducing them to the universal characteristic of rational decision-making. This characteristic, given its deterministic nature, can be conveniently represented in a formal model (Barnes, 1988). How-

ever, the complexity involved in representing economic problems via a formal model escalates when predetermined rules for problem-solving need to be incorporated.

In his groundbreaking study on the claims problem, O’Neill (1982) introduced a formal model for resolving a rights conflict issue. The proposed model drew inspiration from the concept of rights arbitration in labor law, where conflicts are resolved according to established rules and customs. Additionally, the author explored interest arbitration, an alternative dispute resolution method employed in the absence of preexisting rules, where decisions are based on the respective interests of the parties involved.

As previously discussed, Islamic inheritance law is predominantly governed by well-defined mandatory rules. Consequently, any model devised to treat an Islamic law dispute as an economic problem must comply with the relevant Islamic legal rules. The literature presents various approaches to modeling an Islamic law dispute as an economic problem (Chapra, 1996). A group of Islamic scholars resolves economic issues by developing Islamic objectives, another by studying the decision-making behavior of Islamic individuals, and a third by solving economic issues within an Islamic framework, as categorized by Furqani (2018). This paper models the Islamic inheritance law problem in harmony with the perspectives of Haneef (1997), Siddiqui (1976), and Iqbal and Lewis (2009) concerning the resolution of economic problems in an Islamic context. This approach generates solutions that align with the values and norms rooted in the sources of knowledge in Islam.

To illustrate our formal model, consider a fair division problem involving a divisible resource, denoted as $E \in \mathbb{R}_+$, which needs to be allocated among a finite set of individuals denoted as $N = \{1, 2, \dots, n\} \in \mathbb{N}$. Each individual $i \in N$ has a claim, denoted as $c_i \in \mathbb{R}_+$, on the resource, and the vector $c = (c_i)_{i \in N}$ represents the profile of individuals’ claims.

The triple $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$ represents a claims problem if the total claims of individuals exceed the available resource, $\sum_{i \in N} c_i \geq E$. The core challenge of the claims problem lies in the inability to satisfy all claimants simultaneously due to excess demand. In contrast, the triple $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$ represents a surplus-sharing problem if the total claims of individuals are less than or equal to the available resource, $\sum_{i \in N} c_i \leq E$. The essence of the surplus-sharing problem lies in the distribution of excess resources after satisfying all claimants.

The aim is to determine a solution that assigns a list of awards to each claimant, ensuring the sum of awards equals the available resource. The set of awards for a problem (N, c, E) is denoted as $X(N, c, E)$. A division rule, denoted by S , is a function that assigns an awards vector from $X(N, c, E)$ to each problem (N, c, E) . We will discuss the most notable fair division rules below and apply these rules to the cases examined in subsequent sections.

Proportionality has long been the foremost method for solving simple division problems, as recorded in history. Aristotle, often cited in this context, viewed proportionality as equity. The proportional rule in fair division problems, therefore, ensures each claimant receives a portion of the available resources in proportion to their claim. This rule reflects Aristotle’s notion that equals should be treated equally, and unequals unequally, by distributing resources proportional to the size of each claim, thus attaining

a sense of justice and equity. For Claims and surplus-sharing Problems, the Proportional (P) Rule is formulated as:

Proportional (P) Rule:

For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$,

$$P_i(N, c, E) = \lambda c_i \text{ for all } i \in N$$

where $\lambda = \frac{E}{\sum_{i \in N} c_i}$.

As the formula suggests, the Proportional (P) Rule assigns a list of "awards" to each claimant such that the sum of their awards equals the resource. This is accomplished by multiplying each claim by λ which is calculated as the available resource divided by the sum of the claims on the resource.

We now transition from equal division based on equality per unit of claim to equal division in absolute terms. The ensuing rule retains the concept of equal division provided that no individual receives more than their claim. This rule, advocated by numerous medieval scholars such as Maimonides, is the Constrained Equal Awards (CEA) Rule and is formulated for Claims Problems as follows:

Constrained Equal Awards (CEA) Rule:

For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$,

$$CEA_i(N, c, E) = \min\{c_i, \lambda\} \text{ for all } i \in N$$

where $\lambda \in \mathbb{R}_+$ is such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

The next rule, akin to the constrained equal awards rule, promotes the concept of equality. However, it takes a different approach by focusing on the losses experienced by claimants rather than what they receive. This rule aims to make losses as equal as possible while ensuring no one receives a negative amount. This rule also features in Maimonides' writings. The Constrained Equal Losses (CEL) Rule is defined in the context of Claims Problems as follows:

Constrained Equal Losses (CEL) Rule:

For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$,

$$CEL_i(N, c, E) = \max\{c_i - \lambda, 0\} \text{ for all } i \in N$$

where $\lambda \in \mathbb{R}_+$ is such that $\sum_{i \in N} \max\{c_i - \lambda, 0\} = E$.

The next rule is defined for scenarios with two claimants and is based on the principle that if one claimant's claim is less than the total estate to be divided, the uncontested portion of the estate is effectively conceded to the other claimant. In this approach, each claimant is initially allocated the amount conceded to them by the other claimant. The remaining amount is then divided equally between the two claimants. In the context of Claims Problems, this rule is formally known as the

Concede-and-Divide Rule (CD) and can be stated as follows:

Concede-and-Divide Rule (CD):

Let $n = 2$. For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^2 \times \mathbb{R}_+$, we define the CD rule as follows:

$$CD_i(N, c, E) = \left(\max(E - c_j, 0) + \frac{E - \max(E - c_j, 0) - \max(E - c_i, 0)}{2} \right) \text{ for all distinct } i, j \in N$$

The last rule we will define is indeed a generalization of the Concede-and-Divide Rule (CD) to accommodate scenarios with more than two claimants. In the context of Claims Problems, the Ibn Ezra (IE) Rule is formally stated as:

Ibn Ezra (IE) Rule:

For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$, we order the claims from the largest to the smallest as $c_1 \geq c_2 \geq \dots \geq c_{n-1} \geq c_n$. Then, we define the IE rule as follows:

$$\begin{aligned} IE_n(N, c, E) &= \frac{c_n}{n}, \\ IE_{n-1}(N, c, E) &= \frac{c_n}{n} + \frac{c_{n-1} - c_n}{n-1}, \\ &\vdots \\ IE_1(N, c, E) &= \frac{c_n}{n} + \frac{c_{n-1} - c_n}{n-1} + \dots + \frac{c_2 - c_3}{2} + c_1 - c_2. \end{aligned}$$

3.1 Claims Problems Adaptations

There are three distinct scenarios that can arise from the relationship between the total claims' sizes and the available resources. These potential scenarios are demonstrated in Figure 3 for instances with two heirs.

The first scenario occurs when $\sum_{i \in N} c_i > E$. This is referred to as Awliyya in Islamic inheritance law. Legal disputes that can be represented by the triple $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$ can be modeled as claims problems in this case.

In the second scenario, where $\sum_{i \in N} c_i = E$, it is known as Adliyya in Islamic inheritance law. Disputes under this case cannot be modeled as claims or surplus-sharing problems, as there is no problem to solve; all claims can be satisfied with the available resources.

The third scenario, called Raddiyya in Islamic inheritance law, occurs when $\sum_{i \in N} c_i < E$. Legal disputes that can be represented by the triple $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$ in this scenario can be modeled as surplus-sharing problems.

The following three Awliyya cases from Islamic history are modeled as claims problems. The solutions proposed by Ali, Shuraih al-Qadhi, 'Umar, and Ibn 'Abbas are presented, which are subsequently

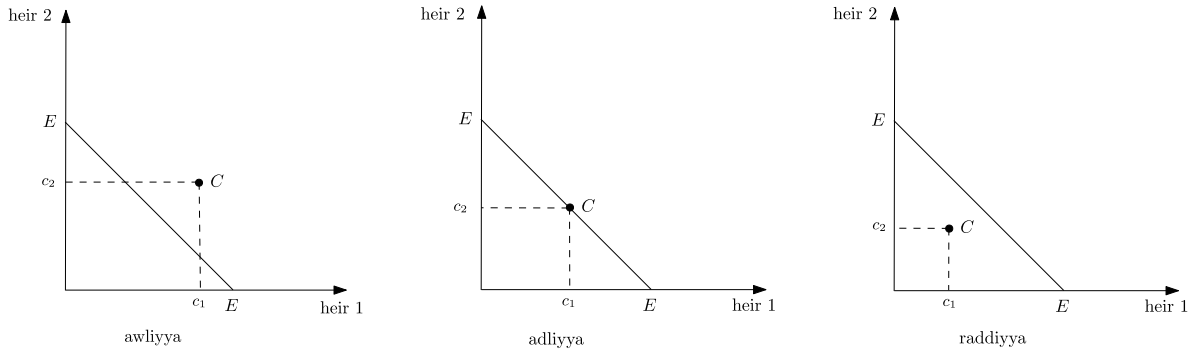


Figure 3: Generic Islamic cases for two heirs

recognized as sharing methods. These are explored in conjunction with solutions derived from the application of classic fair division rules. These rules ensure that a list of awards is allocated to each claimant, such that the total of their awards equates to the available resources.

3.1.1 al-Minbariyya or the Case of Pulpit

The case is named pulpit as it arose when Ali was interrupted while giving a sermon from the pulpit (Abdulla & Keshavjee, 2018). According to the case, a man dies leaving a wife (W), two daughters (DS), and both of his parents (PS), as cited in (Ajetunmob, 1988; Al-Sarakhsi, 1087b; Coulson, 1971; Jones & Rumsey, 1890; Seydisehri, 1883; Tyabji, 1940). By law, the legal shares are $\frac{1}{8}$, $\frac{2}{3}$, and $\frac{1}{3}$ respectively, with the total of these shares or legal claims equaling $\frac{27}{24}$. Therefore, the estate cannot compensate all claimants. In Table 2, in addition to the solution proposed by Ali (ALI), which is consistent with the ‘Awl doctrine, solutions suggested by the classic fair division rules are presented.

Ajetunmob (1988), Cilaro (2012), Coulson (1971), Jones and Rumsey (1890), and Tyabji (1940) provide legal reasoning by describing Ali’s method as a proportional diminution of the shares.

Table 2: al-Minbariyya or the Case of Pulpit

Heir	Claim	ALI	P	CEA	CEL
W	1/8	1/9	1/9	1/8	1/12
DS	2/3	16/27	16/27	13/24	5/8
PS	1/3	8/27	8/27	1/3	7/24
Total	27/24	1	1	1	1

3.1.2 al-Shuraihiyya or the Case of Shuraih al-Qadhi

al-Shuraihiyya is an Awliyya case from the era of Shuraih al-Qadhi, a judge who served under different caliphs. In this case, a woman dies leaving a husband (H), two full sisters (FSS), two uterine sisters (USS), and a mother (M) as cited in (Ajetunmob, 1988; Al-Sarakhsi, 1087b; Coulson, 1971; Jones &

Rumsey, [1890]; Seydisehrî, [1883]; Tyabji, [1940]) By law, the legal shares are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{3}$, and $\frac{1}{6}$ respectively, with the total of these shares or legal claims equaling $\frac{10}{6}$. Therefore, the estate cannot satisfy all claimants. In Table 3, in addition to the solution proposed by Shuraih al-Qadhi (SHU), which is consistent with the ‘Awl doctrine, solutions suggested by the classic fair division rules are presented.

Table 3: al-Shuraihiyya

Heir	Claim	SHU	P	CEA	CEL
H	1/2	3/10	3/10	5/18	1/3
FSS	2/3	4/10	4/10	5/18	1/2
USS	1/3	2/10	2/10	5/18	1/6
M	1/6	1/10	1/10	1/6	0
Total	10/6	1	1	1	1

Al-Sarakhsi ([1087b]) and Jones and Rumsey ([1890]) quote that Shuraih al-Qadhi provides legal reasoning for his distribution by stating that his decision is align with the judgment of ‘Umar on the same case.

3.1.3 al-Mubahala

Al-Mubahala, which refers to the invocation of God’s curse, is an Awliyya case from the era of ‘Umar as cited in (Ajetunmob, [1988]; Al-Sarakhsi, [1087b]; Coulson, [1971]; Jones & Rumsey, [1890]; Seydisehrî, [1883]; Tyabji, [1940]). The case involves a woman who passed away, leaving behind a husband (H), a mother (M), and a full sister (FS). The respective legal claims for each party are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{2}$.

According to ‘Umar (UM) ruling, the husband receives $\frac{3}{8}$, the mother receives $\frac{2}{8}$, and the sister receives $\frac{3}{8}$. On the other hand, Ibn ‘Abbas (AB) suggests an alternative allocation where the husband receives $\frac{1}{2}$, the mother receives $\frac{1}{3}$, and the sister receives $\frac{1}{6}$.

The Proportional (P) Rule indicates that the husband receives $\frac{3}{8}$, the mother receives $\frac{2}{8}$, and the sister receives $\frac{3}{8}$, which is consistent with ‘Umar’s ruling. In Table 4, in addition to the solution proposed by ‘Umar (UM) and Ibn ‘Abbas (AB), solutions suggested by the classic fair division rules are presented.

Table 4: al-Mubahala

Heir	Claim	UM	AB	P	CEA	CEL
H	1/2	3/8	1/2	3/8	1/3	7/18
M	1/3	2/8	1/3	2/8	1/3	4/18
FS	1/2	3/8	1/6	3/8	1/3	7/18
Total	4/3	1	1	1	1	1

Corollary 1.

The doctrine of ‘Awl, applicable in cases of Awliyya, defined herein as claims problems, is identical to the Proportional Rule.

The Proportional Rule can be characterized through various sets of axioms, as previously noted. Young (1988) characterizes this rule employing the axioms of self-duality and composition up. The use of these axioms also allows us to characterize the doctrine of ‘Awl. The relevance of these axioms to the Islamic legal context is explained below.

Self-duality, an axiom originally derived from the Talmud according to the fair division literature, symmetrically addresses both gains and losses. This axiom also features as Article no. 88 in Majalla (Ahmad Cevdet Pasha, 1877), the Ottoman Empire’s civil code, fundamentally rooted in Hanafi law. This code states, “The burden is proportional to the benefit and the benefit to the burden”. Khadduri (1953-1954) interprets this article as an instance of objective responsibility, paralleling the Romans’ maxim, “qui sentit commodum, sentire debet et onus et contra (he who enjoys the benefit ought also to bear the burden and vice versa),” as cited in (Lowell, 1895). Conversely, Hocaeminefendizade (2020) elucidates this article’s relevance to the symmetric treatment of gains and losses. For Awliyya cases, represented by $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$, a dual rule of S denoted by S^d is defined as $S_i^d(N, c, E) = c_i - S_i(N, c, \sum_{i \in N} c_i - E)$ for all $i \in N$. Evidently, the Constrained Equal Awards (CEA) and Constrained Equal Losses (CEL) rules are dual to one another. A rule is *self-dual* when it equals its own dual, i.e., $S^d = S$. This concept of duality extends to the properties of rules, where two properties are dual if a rule adhering to one property guarantees that its dual aligns with the other.

Composition up, an axiom introduced by Young (1988), caters to instances of underestimated estate value. It guarantees that total amounts obtained by claimants from the first and second distributions, resulting from an initial underestimation of the material value of the resource, match the amounts they would receive had the value been correctly calculated. This axiom finds its application in Islamic law disputes, as evidenced in obligation law-related disputes (Al-Marghinani, 1986). For Awliyya cases represented by $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$, if $\sum_{i \in N} c_i \geq E' > E$ and $S(N, c, E') = S(N, S(N, c, E), E')$ holds true, then S satisfies the composition up axiom.

Through the axioms of self-duality and composition up, the solutions proposed by Ali, Shuraih al-Qadhi, ‘Umar for the presented cases can be generalized into a singular sharing method. The characteristics embodied by this method are identical to the axioms that characterize the doctrine of ‘Awl and the Proportionality, recognized as a non-Jewish sharing method (Thomson, 2019). Given the extensive acceptance of the doctrine of ‘Awl across various Islamic sects, this discovery opens up a fresh vantage point for juxtaposing Jewish and Islamic inheritance law. Furthermore, as noted by Moulin (1987) and De Frutos (1999), the proportional rule is the sole rule precluding strategic manipulation, implying that, for Awliyya cases, there exist no alternative solutions offering a unique, non-manipulable result for the legal heirs apart from the doctrine of ‘Awl.

3.2 Surplus-Sharing Problems Adaptations

Raddiyya case can be modeled as a surplus-sharing problem. In the legal disputes presented in this section, the material value of the resource is greater than or equal to the total claims, so the triple $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$ when $\sum_{i \in N} c_i > E$ is referred to as a 'surplus-sharing problem'.

Raddiyya disputes are resolved differently by jurists of the time. Coulson (1971) states that the Hanafi, Hanbali, and Shafi'i schools apply in these circumstances the doctrine of Radd, which requires that the surplus of the inheritance be returned to all the entitled Qur'anic heirs. Uthman suggests that any surplus remaining after the initial distribution mandated by law should be distributed proportionally to the initial legal heirs. Similarly, Ali proposes that any surplus remaining after the initial distribution required by law should be distributed proportionally to the initial legal heirs, excluding the husband and wife. According to Ahmad ibn Hanbal and Abdullah ibn Masud, the surplus should be distributed proportionally among the initial heirs, excluding the husband, wife, daughter, sister, and uterine brother. According to an opinion attributed to Ibn 'Abbas, the surplus should be distributed proportionally among the initial heirs, excluding the husband, wife, and grandmother. According to an opinion attributed to Zaid ibn Thabit, Ibn 'Abbas, and Al-Shafi'i, the surplus should be transferred to the bayt al-mal (or royal treasury).

In addition to the solutions proposed by jurists of the time, we also provide solutions proposed by the Proportional (P) Rule, the Constrained Equal Awards (CEA) Rule, and the Constrained Equal Losses (CEL) Rule, which are adapted to the context of the surplus-sharing problem. The formulation for the Proportional (P) Rule is already given above. In the context of surplus-sharing problem, CEA and CEL are formulated as follows:

Constrained Equal Awards (CEA) Rule:

For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$,

$$CEA_i(N, c, E) = \max\{c_i, \lambda\} \text{ for all } i \in N$$

where $\lambda \in \mathbb{R}_+$ is such that $\sum_{i \in N} \max\{c_i, \lambda\} = E$.

Constrained Equal Losses (CEL) Rule:

For all $(N, c, E) \in \mathbb{N} \times \mathbb{R}_+^n \times \mathbb{R}_+$,

$$CEL_i(N, c, E) = c_i + \frac{E - \sum_{i \in N} c_i}{n} \text{ for all } i \in N$$

3.2.1 The Case of Inheritor Sister and Mother

A person dies leaving a full sister (FS) and a mother (M), as cited in (Al-Sarakhsi, 1087b). By law, the legal shares are 1/2 and 1/3 respectively, with the total of these shares or legal claims equaling 5/6. Therefore, the estate has a surplus after satisfying all legal heirs.

In Table 5, in addition to the solutions proposed by Ali (ALI) and Zaid ibn Thabit (ZT), solutions

Table 5: The Case of Inheritor Sister and Mother

Heir	Claim	ALI	ZT	P	CEA	CEL
FS	1/2	3/5	1/2	3/5	1/2	7/12
M	1/3	2/5	1/3	2/5	1/2	5/12
T	0	0	1/6	0	0	0
Total	5/6	1	1	1	1	1

suggested by the classic division rules are also presented. In this case, the Royal Treasury (T) is included as a legal heir to demonstrate the payment made to it through the relevant distribution.

The Case of Inheritor Daughter and Mother

A person dies leaving a daughter (D) and a mother (M), as cited in (Coulson, 1971). By law, the legal shares are 1/2 and 1/6 respectively, with the total of these shares or legal claims equaling 2/3. Therefore, the estate has a surplus after satisfying all legal heirs.

Table 6: The Case of Inheritor Daughter and Mother

Heir	Claim	RADD	P	CEA	CEL
D	1/2	3/4	3/4	1/2	2/3
M	1/6	1/4	1/4	1/2	1/3
Total	2/3	1	1	1	1

In Table 6, in addition to the solutions proposed by the doctrine of Radd (RADD), solutions suggested by the classic division rules are presented.

In the context of surplus-sharing problems, the Proportional (P) Rule can be characterized by axiom of Independence of Merging (or Splitting) or by Decomposition. (Moulin, 2002). We conjecture that the doctrine of Radd, which is applied in cases of Raddiyya (defined as surplus-sharing problems), is identical to the Proportional Rule. Therefore, it can be characterized through the same axioms.

3.3 Other Fair Division Problems Adaptations

Claims problems and surplus-sharing problems deal with the distribution of resources among claimants with legal claims. Such legal disputes can be modeled as claims or surplus-sharing problems only when the resource has positive material value. Legal disputes failing to meet all criteria for modeling as claims or surplus-sharing problems can still be modeled as adapted versions of these problems.

3.3.1 The Case of Three Heir Sons and The Estate’s Creditor

Al-Sarakhsi (1087b) presents a legal dispute in which a man dies, leaving three sons. The net estate, valued at 3000 dirhams, is equally distributed among the sons. However, a creditor later claims that the deceased father owed him 3000 dirhams. The oldest son acknowledges the entire debt, the middle son admits to 2000 dirhams, and the youngest son concedes 1000 dirhams. If their claims are accepted, the sons should pay 3000/3 dirhams, 2000/3 dirhams, and 1000/3 dirhams respectively. Yet, these payments amount to only 2000 dirhams, insufficient to meet the creditor’s claim of 3000 dirhams.

Abu Yusuf and Al-Shaybani offer different solutions and justifications for settling the dispute. Table 7 presents the legal shares, concessions of the sons, and the amount they should pay to the creditor according to the solutions proposed by Abu Yusuf (AY) and Muhammad Al-Shaybani (MS). Additionally, the solutions obtained by applying Ibn Ezra’s (IE) Method and the Constrained Ibn Ezra’s (CIE) Method are also presented. The latter method is identified by introducing a constraint for the Ibn Ezra (IE) Method.

Table 7: The Case of Three Heir Sons and The Estate’s Creditor

Heir	Share	Concession	AY	MS	IE	CIE
Eldest Son	1000	3000	1000	1000	5500/3	1000
Middle Son	1000	2000	2500/3	1000	2500/3	2500/3
Youngest Son	1000	1000	1000/3	1000/3	1000/3	1000/3

Abu Yusuf justifies his method of distribution by considering the amounts each son concedes. Conversely, Al-Shaybani’s method accounts for the debt each son acknowledges and their respective inheritance shares. For Abu Yusuf, the sons’ payments to the creditor align with their concessions: all sons concede 1000 dirhams collectively, the middle and eldest sons concede 2000 dirhams collectively, and only the eldest son concedes the full 3000 dirhams. Therefore, each son initially pays 1000/3 dirhams from their inheritance share, as each son collectively concedes 1000 dirhams. After this payment, the youngest son is free of debt, having conceded and paid for only 1000 dirhams of the total debt. The middle and eldest sons, who concede an additional 1000 dirhams of the outstanding debt, pay an extra 1000/2 dirhams from their shares. After this payment, the middle son is also debt-free, having conceded and paid for his portion of the 2000 dirhams debt. The remaining 1000 dirhams, conceded solely by the eldest son, should be paid entirely by him. Hence, the youngest son must pay $\frac{1000}{3}$ dirhams, the middle son pays $(\frac{1000}{3} + \frac{1000}{2})$ dirhams, and the eldest son pays $(\frac{1000}{3} + \frac{1000}{2} + \frac{1000}{1})$ dirhams. As the calculated amount for the eldest son exceeds his inheritance share, he is required to pay only 1000 dirhams.

Al-Shaybani, on the other hand, asserts that the eldest son should first relinquish his entire inheritance share (1000 dirhams) to the creditor, given his full acknowledgment of the debt. Since the middle son concedes 2000 dirhams of the debt and the eldest son has already paid half of it, the middle son should pay the creditor the remaining 1000 dirhams. Lastly, the youngest son should pay 1000/3 dirhams, acknowledging only 1000 dirhams of the debt, with 1000/3 dirhams as his share of this portion of the

debt.

The problem cannot be modeled as a surplus-sharing problem because there is no surplus of the resource to allocate among the claimants. The resource is fixed (i.e., the remaining debt owed by the father to the creditor), and the question is how to allocate it among the claimants in a fair manner. A claims problem arises when there is a dispute over how to allocate a given amount of a divisible resource among multiple claimants who have competing claims on the resource. The problem cannot be modeled as either a claims problem in the classic sense because there is a single claim on the estate which has already been divided among the heirs and there are three distinct counterclaims regarding the proportion of the estate that should remain for debt payment. From a different perspective, the problem can be described as a problem of dividing a single liability among multiple parties. In this case, the problem can be modeled as a modified claims problem by identifying the remaining debt owed by the father to the creditor as the resource, and the three sons who have acknowledged portions of the debt as the claimants.

Accordingly, the problem can be formally modeled as the triple $(N, T, E) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+^n$, where $n = 3$ represents the number of legal heirs, E represents the concessions vector and T represents the material value of the estate. If the original case at hand is modeled to make a generalized legal dispute, the estate (T) is divided among the legal heirs based on their equal inheritance shares ($T/3$) according to the primary sources of Islamic law. Then, when a creditor claims complete ownership of the estate (T), the heirs acknowledge the debt to differing levels. Although the case can be easily generalized, we equalize $T = E_1$ for the sake of simplicity. To apply Ibn Ezra's (IE) method and Abu Yusuf's (AY) method, the concessions are sorted from the largest to the smallest as $T = E_1 \geq E_2 \geq E_3$.

If Abu Yusuf's (AY) Method of distribution is generalized for all possible $n = 3$ cases, the legal heirs should pay $AY_3 = \min\{T/3, E_3/3\}$ and $AY_2 = \min\{T/3, (E_3/3 + (E_2 - E_3)/2)\}$ and $AY_1 = \min\{T/3, (E_3/3 + (E_2 - E_3)/2 + E_1 - E_2)\}$ based on the justification provided by Abu Yusuf as cited in (Al-Sarakhsi, 1087b).

Ibn Ezra's (IE) Method, which is very similar to the procedure applied by Abu Yusuf (AY), can be applied to the case to test the similarities between these methods. As suggested by Alcalde et al. (2008), if IE is applied to the dispute at hand, the legal heirs should pay $IE_3 = E_3/3$, $IE_2 = (E_3/3 + (E_2 - E_3)/2)$, and $IE_1 = (E_3/3 + (E_2 - E_3)/2 + E_1 - E_2)$. This shows that the only difference between IE and AY is that Abu Yusuf ensures legal heirs are responsible for no more than their inheritance shares by defining a constraint. If this constraint is combined with the Ibn Ezra's Method, the Constrained Ibn Ezra's (CIE) Method is developed, and according to CIE, the legal heirs should pay $CIE_3 = \min\{T/3, E_3/3\}$, $CIE_2 = \min\{T/3, (E_3/3 + (E_2 - E_3)/2)\}$, and $CIE_1 = \min\{T/3, (E_3/3 + (E_2 - E_3)/2 + E_1 - E_2)\}$. This shows that the generalization of Abu Yusuf's (AY) method of distribution for all ($n = 3$) cases and the Constrained Ibn Ezra's (CIE) Method are identical.

Corollary 2. Constrained Ibn Ezra's (CIE) Method and Abu Yusuf's (AY) Method coincide for $n = 3$.

Ibn Ezra's method is characterized by Alcalde et al. (2008) through two axioms: anonymity and core-

transition responsiveness. Conversely, Thomson (2019) characterizes the same method using three axioms, which include anonymity, claims truncation invariance, and core-transition responsiveness. The use of these axioms, when coupled with the constraint of limited concession, allows for the characterization of both the Constrained Ibn Ezra’s Method (*CIE*) and Abu Yusuf’s Method (*AY*). The relevance of these axioms and the constraint to the Islamic legal context is further explained below.

The axiom of anonymity requires that the identities of the participants should not influence resource allocation. This axiom resonates with Islamic law disputes, reflecting the principle of justice in Islam. Islam emphasizes the importance of treating individuals fairly and impartially, regardless of their social status, identity, or qualifications. It advocates for the principle that all individuals should have equal rights and opportunities to access resources, receive their due entitlements, and be treated justly. This equitable principle is profoundly emphasized in the Qur’an, which states: ”O you who have believed, be persistently standing firm in justice, witnesses for Allah, even if it be against yourselves or parents and relatives. Whether one is rich or poor, Allah is more worthy of both. So follow not [personal] inclination, lest you not be just.” (Qur’an 4:135). This verse clearly illustrates the Quran’s insistence on equity and fairness, asserting that justice must prevail irrespective of an individual’s personal ties, wealth, or status. In the context of the modified claims problem where there are three legal heirs $n = 3$ with equal inheritance shares, and the estate (T) is divided among them based on these equal shares, a distribution rule S satisfies the axiom if $S_i(N, T, E) = S_j(N, T, E)$ holds for any $i, j \in N$ where $E_i = E_j$. This implies that, regardless of their individual identities, if two heirs have an equal amount of concessions, they are required to pay the same amount.

Claims truncation invariance is an axiom introduced by Aumann and Maschler (1985) and later formally modeled by Dagan and Volij (1993). This axiom asserts that no heir should be required to make a concession greater than the value of the estate. In cases where a legal heir’s concession surpasses the estate’s value, the excess is trimmed such that the concession equals the estate’s worth. This axiom’s relevance extends to disputes in Islamic law, as it corresponds closely with the principles quoted by Al-Sarakhsi (1087a) in “Al-Mabsut Volume 18”: “The validity of an heir’s acknowledgment of debt with regard to the deceased is restricted to the value of the estate in their possession.” The application of which is quoted by Uluçay and Abdullah (2016) “Severe damage is made to disappear by a lighter damage.” These reflect the principle of claims truncation invariance, indicating the axiom’s direct applicability in inheritance disputes. A distribution rule, denoted as S , satisfies this axiom if it takes $\min\{E_i, T\}$ for each $i \in N$ as input instead of taking the concession E_i .

Core-transition responsiveness ensures that the payment made by an heir, whose claim is less than or equal to the net estate, remains unchanged even when there are increases in the available resources. This concept was initially introduced by Alcalde et al. (2008), and it resonates with the norms and values upheld in Islam. This correlation is demonstrated by the maxims from the Qur’an and Sunna such as, “The norm is that the status quo remains as it was before,” quoted by Kamali (2015), and “When the receiving of a thing is forbidden, the giving of it is also forbidden,” as argued by Uluçay and Abdullah (2016). The applicability of these principles across a broad range of issues is underscored by (Musa, 2014), who notes, “These include the five major qawā’id that are accepted by all schools and are applicable to nearly all the fields of fiqh.” When it comes to modified claims problems, which are

represented by the triple $(N, T, E) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+^n$, a distribution rule, denoted by S , is said to adhere to the axiom if, under the condition that $\max E_i \geq T' > T$, it ensures that $S_j(N, T, E) = S_j(N, T', E)$ for every $i, j \in N$ such that $E_j \leq T$.

The constraint of limited concession is a legal principle that prevents a party from admitting to owing more than their fair share of a debt or obligation. This constraint aligns with the norms and values of Islam. As quoted by Kahf (2019), “There should be no harm and no exchange of harms.” Additionally, Saiti and Abdullah (2016) states, “What is permissible in law cannot be a cause for liability.” Further strengthening this principle, Kızılkaya (2021) affirms, “The fundamental principle is freedom from liability.” In the context of a modified claims problem, when the heirs admit to owing varying degrees of debt (expressed as $T \geq E_1 \geq E_2 \geq E_3$), the axiom demands that the admitted amount cannot exceed their respective inheritance shares. If this constraint coincides with a distribution rule S , the payments are adjusted accordingly. The updated equation becomes $X_i(N, T, E) = \min\{\frac{T}{n}, S_i\}$, where i spans across the set of heirs N and X represents the payments vector. This equation ensures that each heir’s liability is proportionate to their inherited share, upholding the principle of limited concession.

Through the adherence to the axioms of anonymity, claims truncation invariance, core-transition responsiveness, and the constraint of limited concession, the Constrained Ibn Ezra (*CIE*) Method finds its definition and application. The ingenious approach taken by Abu Yusuf in devising his distribution methodology, which took place nearly three centuries before Ibn Ezra, indicates that the Abu Yusuf (*AY*) method, when generalized for $n = 3$, corresponds directly to the *CIE* method.

Given the fact that contemporary fair division literature largely neglects Islamic legal disputes and distribution rules, this observation could potentially unlock new horizons for extensive research on Islamic law. It serves as a gateway to deepen our understanding of the intricate balance between fairness, responsibility, and adherence to religious doctrine. Additionally, these findings may provide fresh insights and practical solutions to present-day issues regarding the fair distribution of resources, in accordance with Islamic law.

4 Conclusion

In this study, we examined selected legal disputes from Islamic law history by modeling them as claims problems, surplus-sharing problems, and modified claims problems. The solutions proposed by authorities and jurists of the time were generalized as distribution rules, characterized by context-specific axioms, and compared axiomatically with classic division rules.

Our investigations revealed that the Doctrine of Awl in Islamic law, applied in Awliyya cases, is identical to the Proportionality Rule within the context of claims problems. Within the framework of surplus-sharing problems, the doctrine of Radd, utilized in Raddiyya cases under Islamic law, mirrors the Proportional Rule.

We introduced the Modified Claims Problem as a novel problem type incorporating the constraint of limited concession. Abu Yusuf’s method was also generalized as a distribution rule within the context

of this newly defined modified claims problem. We established that Abu Yusuf's method aligns with the Constrained Ibn Ezra's Method.

Our findings significantly contribute to the extant literature on fair division and Islamic law by introducing a new type of fair division problem and characterizing new fair division rules through adapted axioms. Remarkably, previous fair division literature has not referenced Islamic legal disputes. Thus, this study may serve as a foundation for further explorations into additional cases.

Historically, literature on Islamic inheritance law has offered historical, social, and legal perspectives to comprehend the logic behind the distributions provided for contentious cases. Our novel study introduces an axiomatic analysis that facilitates a comprehensive normative analysis of complex cases involving Islamic law disputes. This innovative approach allows for a deeper understanding and evaluation of the normative aspects inherent in such cases. Despite the limited number of studies examining controversial disputes, our study underscores the importance of a detailed examination of legal cases and adds value to the existing literature.

Additionally, there is a scarcity of studies that compare Jewish and Islamic inheritance laws, examining their similarities, differences, and underlying principles. While these studies have primarily adopted a socio-historical approach, they have garnered significant attention. This study offers new insights regarding these comparisons. We discovered that a widely accepted de facto allocation in Islam aligns with a rule known as the non-Jewish rule. Interestingly, Abu Yusuf's distribution method predates Ibn Ezra's by over three centuries. This suggests that the rule attributed to Ibn Ezra might have originally stemmed from an Islamic method, thereby opening a new discourse in comparative studies previously focused on determining whether Islamic law principles adopted rules from Jewish law.

References

- Abdel-Wahab, S.-E. (1962). Meaning and structure of law in Islam. *Vanderbilt Law Review*, 115–130.
- Abdulla, R. S., & Keshavjee, M. M. (2018). *Understanding sharia: Islamic law in a globalised world*. Bloomsbury Publishing.
- Ahangar, M. A. H. (2002). Islamic law of inheritance: Some misconceptions in Mimi Kamariah Majid's family law in malaysia. *IUMLJ*, 10, 219.
- Ahmad Cevdet Pasha. (1877). *The civil code of the Ottoman Empire: Majalla: [English Translation of Al-Majalla al-Ahkam al-Adaliyyah: Mjalla/Mecelle]*. Ottoman Empire.
- Ajetunmob, M. A. (1988). Collection and review of cases of al-'awl (pro-rata reduction) in shari'ah law of succession. *Islamic Studies*, 209–219.
- Akhlaq, S. H. (2023). *Understanding 'sectarianism': Sunni-Shi'a relations in the Modern Arab World* [Review of the work by Fanar Haddad, New York, Oxford University Press, 2022, viii+368 pp., No Price (hardback), ISBN 9780197510629]. Oxford University Press.
- Alcalde, J., Marco, M. d. C., & Silva, J. A. (2008). The minimal overlap rule revisited. *Social Choice and Welfare*, 109–128.
- Ali, S. S. (2016). *Modern challenges to Islamic law*. Cambridge University Press.
- Al-Marghinani, B. D. (1986). *Al-hidayah* (A. Meylani, Trans.). Darul Ishaat.
- Al-Sarakhsi, M. b. A. (1087a). *Al-mabsut vol. 18* (M. C. Akşit, Trans.; Vol. 28). Gümüşev Yayıncılık.
- Al-Sarakhsi, M. b. A. (1087b). *Al-mabsut vol. 28* (M. C. Akşit, Trans.; Vol. 28). Gümüşev Yayıncılık.
- Anderson, J. N. D. (1965). Recent reforms in the Islamic law of inheritance. *International & Comparative Law Quarterly*, 349–364.
- Arikewuyo, N. A. (2023). Literalism versus rationalism in the house of Islam: A case study of Islamic law of succession. *International Journal of Fiqh and Usul al-Fiqh Studies*, 7(1), 57–67.
- Aumann, R. J., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, 195–213.
- Barnes, T. J. (1988). Rationality and relativism in economic geography: An interpretive review of the homo economicus assumption. *Progress in Human Geography*, 12(4), 473–496.
- Bergantiños, G., & Moreno-Terner, J. D. (2020). Sharing the revenues from broadcasting sport events. *Management Science*, 2417–2431.
- Berkah, D., & Sawarjuwono, T. (2019). Inheritance wealth distribution model and its implication to economy. *Humanities & Social Sciences Reviews*, 7(3), 01–10.
- Bowen, J. (1998). "You May Not Give It Away": How Social Norms Shape Islamic Law in Contemporary Indonesian Jurisprudence. *Islamic Law and Society*, 5(3), 382–408.
- Chapra, U. (1996). *What is Islamic economics?* Islamic Research; Training Institute.
- Cheema, S. A. (2012). Shia and Sunni laws of inheritance: A comparative analysis. *Pakistan Journal of Islamic Research*, 10. <https://doi.org/10.xxxx/pjir.2012.10>
- Cilardo, A. (Ed.). (2012). *The early history of Ismaili jurisprudence: Law under the Fatimids*. Bloomsbury Publishing.
- Coulson, N. J. (1971). *Succession in the Muslim family*. Cambridge University Press.
- Csóka, P., & Herings, P. J.-J. (2021). An axiomatization of the proportional rule in financial networks. *Management Science*, 67(5), 2799–2812. <https://doi.org/10.1287/mnsc.2020.3760>

- Dagan, N. (1996). New characterizations of old bankruptcy rules. *Social Choice and Welfare*, 51–59.
- Dagan, N., & Volij, O. (1993). The bankruptcy problem: A cooperative bargaining approach. *Mathematical Social Sciences*, 287–297.
- De Frutos, M.-A. (1999). Coalitional manipulations in a bankruptcy problem. *Review of Economic Design*, 255–272.
- Du Plessis, P. J. (2015). *Borkowski's textbook on Roman law*. Oxford University Press.
- Eich, T. (2012). Constructing kinship in Sunni Islamic legal texts. In *Islam and assisted reproductive technologies: Sunni and Shia perspectives* (p. 27).
- Engle, E. (2012). The history of the general principle of proportionality: An overview. *Dartmouth Law Journal*, 1–12.
- Epstein, I. (1935). *The babylonian talmud*. Soncino Press.
- Ertemel, S., & Kumar, R. (2018). Proportional rules for state contingent claims. *International Journal of Game Theory*, 229–246.
- Furqani, H. (2018). Defining Islamic economics: Scholars' approach, clarifying the nature, scope and subject-matter of the discipline. *Turkish Journal of Islamic Economics*, 69–93.
- Gandz, S. (1938). The algebra of inheritance: A rehabilitation of al-Khuwarizmi. *Isis*, 319–391.
- Ghafar Ismail, A., Taufiq Possumah, B., & Najib Abdul Kadir, M. (2014). Inter-generational transfer under Islamic perspective. *Humanomics*, 30(2), 95–121.
- Ginsburgh, V., & Zang, I. (2001). Sharing the income of a museum pass program. *Museum Management and Curatorship*, 19(4), 371–383.
- Gupta, A. (2021). Inheritance laws: An analysis. *Jus Corpus LJ*, 2, 688.
- Haneef, M. (1997). Islam, the Islamic worldview, and Islamic economics. *IIUM Journal of Economics and Management*, 5(1), 39–66.
- Hazleton, L. (2010). *After the Prophet: The epic story of the Shia-Sunni split in Islam*. Anchor.
- Heo, E. J., & Lee, J. (2023). Allocating CO2 emissions: A dynamic claims problem. *Review of Economic Design*, 27(1), 163–186. <https://doi.org/10.1007/s10058-022-00269-2>
- Hirsch, A. (2004). Default rules in inheritance law: A problem in search of its context. *Fordham Law Review*, 73(3), 1031–1101.
- Hocaeminefendizade, A. H. (2020). *Dürerü'l-hükkâm şerhu mecelleti'l-ahkâm* (Vol. 1). Diyanet İşleri Başkanlığı Yayınları.
- Holtmann, P. (2014). A primer to the Sunni-Shia conflict. *Perspectives on Terrorism*, 8(1), 142–145.
- Iqbal, Z., & Lewis, M. (2009). *An Islamic perspective on governance*. Edward Elgar Publishing.
- Islam, S. (2019). Development of Muslim law of succession. *International Journal of Management, Technology, and Social Sciences (IJMTS)*, 4(1).
- Janin, H., & Kahlmeyer, A. (2015). *Islamic law: The sharia from Muhammad's time to the present*. McFarland.
- Jones, W., & Rumsey, A. (1890). *Al Sirajiyah, or, the Mahommedan law of inheritance*. Thacker, Spink. <https://www.loc.gov/item/ltf91027047>
- Kahf, M. (2019). Methodology of Islamic economics. In N. Kızılkaya (Ed.), *Islamic economics' methodology and fiqh* (pp. 161–180). Routledge.

- Kamali, M. H. (2015). *Qawa 'id al-fiqh: The legal maxims of Islamic law*. The Association of Muslim Lawyers.
- Kelsen, H. (1959-1960). What is the pure theory of law. *Tulane Law Review*, 33(4), 269–276.
- Khadduri, M. (1953-1954). Nature and sources of Islamic law. *George Washington Law Review*, 21(1), 3–23.
- Kızılkaya, N. (2021). *Legal maxims in Islamic law: Concept, history and application*. Brill - Nijhoff.
- Kuran, T. (1995). Islamic economics and the Islamic subeconomy. *Journal of Economic Perspectives*, 9(4), 155–173.
- Littlechild, S. C., & Owen, G. (1973). A simple expression for the shapley value in a special case. *Management Science*, 20(3), 370–372.
- Lowell, E. (1895). General average: "qui sentit commodum sentire debet et onus". *Harvard Law Review*, 185–197.
- Moulin, H. (1985). Egalitarianism and utilitarianism in quasi-linear bargaining. *Econometrica: Journal of the Econometric Society*, 49–67.
- Moulin, H. (1987). Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory*, 161–186.
- Moulin, H. (2002). Axiomatic cost and surplus sharing. In *Handbook of social choice and welfare, volume 1* (pp. 289–357). Elsevier.
- Musa, K. (2014). Legal maxims as a genre of Islamic law: Origins, development and significance of al-qawaid al-fiqhiyya. *Islamic Law and Society*, 21(4), 325–365.
- Naqvi, A. R. (1972). *Succession in the Muslim family*.
- O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 345–371.
- Pathak, P. A., Sönmez, T., Ünver, M. U., & Yenmez, M. B. (2021). Fair allocation of vaccines, ventilators and antiviral treatments: Leaving no ethical value behind in health care rationing. *Proceedings of the 22nd ACM Conference on Economics and Computation*, 785–786.
- Powers, D. S. (1993). The Islamic inheritance system: A socio-historical approach. *Arab Law Quarterly*, 8(1), 13–29.
- Pulido, M., Sánchez-Soriano, J., & Llorca, N. (2002). Game theory techniques for university management: An extended bankruptcy model. *Annals of Operations Research*, 129–142.
- Rasban, S., Abdullah, A., & Hasan, A. (2020). An analysis of residue net estate distribution to bayt al-māl in Singapore. *ISRA International Journal of Islamic Finance*.
- Rohman, A. N. (2022). Shifting the role of mediation in Islamic inheritance disputes: An overview of Islamic legal philosophy. *Diponegoro Law Review*, 7(2), 230–244.
- Saiti, B., & Abdullah, A. (2016). The legal maxims of Islamic law (excluding five leading legal maxims) and their applications in Islamic finance. *Journal of King Abdulaziz University: Islamic Economics*, 139–151.
- Schacht, J. (1979). *The Origins of Muhammadan Jurisprudence*. Oxford at the Clarendon Press.
- Seydisehrî, M. (1883). *Ferâidü'l-ferâiz* (I. Uca, Trans.). Istanbul.
- Shania. (2021). Muslim law of inheritance: Hanafi school. *Jus Corpus LJ*, 2, 725.
- Siddiqui, A. H. (1976). *Sahih Muslim*. Peace Vision.

- Sitkoff, R. H., & Dukeminier, J. (2017). *Wills, trusts, and estates*. Aspen Publishing.
- Souaiaia, A. (2005). On the sources of Islamic law and practices. *Journal of Law and Religion*, 21, 123–147.
- Taqiyuddin, H. (2021). The justice dimensions in the provision of al-'awl on Islamic inheritance law. *Journal for Integrative Islamic Studies*, 7(2), 135–146.
- Thomson, W. (2019). *How to divide when there isn't enough*. Cambridge University Press.
- Tyabji, F. B. (1940). *Muhammadan law*. N. M. Tripathi & Co.
- Uluçay, B., & Abdullah, A. (2016). The legal maxims of Islamic law (excluding five leading legal maxims) and their applications in Islamic finance. *Journal of King Abdulaziz University: Islamic Economics*, 29(2).
- van der Waerden, B. L. (2013). *A history of algebra: From al-Khwārizmī to Emmy Noether*. Springer Science & Business Media.
- Yazıcı, A. (2013). Fıkıhta meşhur bir miras meselesi olarak avliyye: Avl halleri, kaynağı ve çözümlerinin değerlendirilmesi. *Journal of Faculty of Theology of Istanbul University/Istanbul Üniversitesi İlahiyat Fakültesi Dergisi*, 29.
- Young, H. P. (1987). On dividing an amount according to individual claims or liabilities. *Mathematics of Operations Research*, 12(3), 398–414.
- Young, H. P. (1988). Distributive justice in taxation. *Journal of Economic Theory*, 44(2), 321–335.