Sequential School Choice with Public and Private Schools*

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Abstract

Motivated by school admissions in Turkey and Sweden, we investigate a sequential two-stage admission system with public and private schools. A sequential notion of truthfulness, called straightforwardness, is introduced. Contrary to one-stage systems, sequentiality leads to a trade-off between the existence of a straightforward equilibrium and non-wastefulness. We identify the unique set of rules for two-stage systems that guarantees the existence of a straightforward equilibrium and reduces waste. Existing admission systems are analyzed within our general framework.

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1 Introduction

In most countries, both private and public schools are an integral part of the education system. In the OECD countries, for example, 18 percent of all 15-year-old students are admitted to privately

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managed schools. In fact, among the OECD countries and their 14 partner economies, only Azerbaijan, Romania and the Russian Federation do not have any private schools (OECD, 2012).

In many of these countries, school seats are sequentially assigned to students in two or more stages. The main difference between one-stage and multi-stage admission systems is that the latter type of system must include a set of rules that determine, e.g, the subset of schools that are available in each stage, the set of students that are allowed to participate in each stage, and a stage-dependent mechanism for determining school assignments. In, for example, New York City and Boston the admissions to exam schools and regular public schools are conducted separately and sequentially from each other, and students are allowed to select at most one school from the two separate admissions.[1] In the context of high school admissions to private and public high schools in Turkey, the order in which public and private schools admit students and whether a student can vie for both types of schools without commitment to enroll has changed in recent years. A third example is the Greater Stockholm Region in Sweden, where a centralized public school assignment follows the admissions to private schools (Kessel and Olme, 2018). In fact, in almost all of the 290 Swedish municipalities, the assignment to private and public schools is either parallel or sequential.[2] An example outside of school choice is college admissions in China which are based on a tiered-admissions system where assignments to prestigious top-tier colleges are followed by assignments to lower-tier colleges (Chen and Kesten, 2017).

The pioneering work by Abdulkadiroğlu and Sönmez (2003) initiated a large literature on centralized school choice. The vast majority of the papers in this tradition has restricted their attention to a stylized setting where students are assigned to public schools in a *single* admission stage.[3] In particular, much attention has been directed towards the trade-off between stability and efficiency while maintaining good incentive properties (see, e.g., Abdulkadiroğlu et al., 2009; Kesten, 2010). However, systems where school choice is *de facto* if not *de jure* sequential is the norm rather than the exception. The main characteristic that distinguishes the considered framework from most of the existing literature is that our framework focuses on multi-stage admission systems that may feature any possible combination of centralized and decentralized admissions, which is a prevalent feature of almost all real-life student admissions.

When students seek assignment through sequential admissions, presumed properties possessed under static admission systems may be compromised. This is because students need to make multiple decisions over time, which inevitably leads to spillovers across different types of

[1]For an early account of the system in New York City, see Abdulkadiroğlu et al. (2009) who also hinted at the problems studied in the current paper. The system in Boston has recently undergone a series of reforms. For a detailed description of the current system, see `www.bostonpublicschools.org/Page/7080` and `www.bostonpublicschools.org/Page/6594` (Last accessed: 7/29/2022).

[2]Problems associated with such admissions has been heavily debated in Sweden in the last few years, see, e.g., the following op-ed in one of the leading Swedish newspapers (Dagens Nyheter): `https://www.dn.se/debatt/tre-reformer-som-behovs-for-ett-mer-likvardigt-skolval`.

[3]Very few papers exclusively model decentralized college admissions. See, for example, Chade et al. (2014) and Che and Koh (2016).

schools, e.g., a student admitted both at a public and a private school may subsequently vacate her seat at either school leading to a vacancy, which can potentially jeopardize the stability of the assignments obtained through a one-shot admissions system. A central question then concerns the trade-off between welfare and incentive issues. In particular, this paper addresses the following question: how should a multi-stage system be ideally organized in terms of timing, participation, and assignment rules across and within stages in light of such trade-offs? To answer this question, we model a general sequential framework that contains many real-life admission systems as special cases (see Section 5).

Of particular interest for our analysis are systems in which only public schools are considered in the first admission stage and only private schools are considered in the second admission stage or vice versa as in, e.g., Turkey and some municipalities in Sweden (see Section 1.1). Similar to the existing literature, the point of departure is a set of desirable axioms. From the early school choice literature, it is known that some of these axioms are considered to be of particular importance (see, e.g., Abdulkadiroğlu and Andersson, 2023; Abdulkadiroğlu and Sönmez, 2013, for an overview). One such property is non-wastefulness which means that after the matching mechanism has assigned the students to schools, there should exist no student who prefers a school with an empty seat to her assigned school. Another such property is that the matching mechanism should be designed in such fashion that it is impossible for students to gain by strategic misrepresentation of their preferences. Even if most of the considered axioms naturally can be extended to the multi-stage setting (e.g., non-wastefulness), it is more complicated to define a multi-stage notion of truthfulness. To encompass the latter, we define a sequential notion of truthfulness called "straightforwardness." Broadly speaking, this notion means that the students report their true preferences over the (relevant) schools in each round of the sequential admission process. The notion also captures the idea of sequential rationality as it implies that students aim to improve their current situation in each admission round.

In one-stage admission systems, mechanisms that always select a non-wasteful matching where students, in addition, have incentives to truthfully report their preferences are known to exist. A prominent example of such a mechanism is the student-proposing deferred acceptance algorithm (Abdulkadiroğlu and Sönmez, 2003; Gale and Shapley, 1962). It is, however, more demanding to design a matching mechanism that satisfies these two specific properties in multi-stage admission systems due to the dynamic nature of sequential systems. That is, multi-stage systems assign students in each stage and allow students to adopt more complex strategies. As a consequence, school seats may be wasted in each stage of the admission process, and the complex structure of the strategy sets makes it more demanding to exclude the possibility of successful manipulation. Our results also show that there is a trade-off between the existence of a straightforward Subgame Perfect Nash Equilibrium (SPNE) and non-wastefulness. In fact, the complex nature of sequential admission systems leads to that very specific admission rules, related to when students are allowed to participate in specific rounds and whether or not they can "keep" their assignment from the first stage of the admission process when participating in the second stage, are required to guarantee the existence of a straightforward SPNE. Our main result (Theorem 1) states that there is a unique set of rules for sequential assignment systems which guarantees the existence of a straightforward SPNE, respects priorities, and at the same time minimizes the waste of school seats.[4] These results enable us to propose a new system for two-stage school admissions.[5] In particular, Theorem 1 entails a new type of acyclicity restriction on the primitive structure and thereby offers new insights as to how schools can be prioritized to achieve the desiderata. We also establish a revelation principle result for our context.[6] In Section 6, we provide a direct mechanism, called *decomposing-DA*, whose outcome coincides with the straightforward equilibrium outcome under the unique set of rules identified in Theorem 1.

Our results suggest a way to minimize waste while preserving truthful incentives. Importantly, the proposed approach incentivizes private schools to voluntarily join centralized assignment, which in turn facilitates convergence to a unified centralized assignment system that can readily avoid the trade-offs faced in sequential assignments.

1.1 Sequential School Admission Systems

The high school admissions system that determines the placements of over a million students in Turkey has recently undergone a series of reforms and remains a source of much controversy. In the system that was in use in 2014, students who were admitted to public schools in a centralized first stage could go on to be admitted to private schools in a decentralized second stage before eventually deciding on their actual assignments. This "admission without commitment" feature of the system gave rise to tens of thousands of seats of both types of schools to be subsequently vacated. In the aftermath of the epic number of vacancies, the Ministry of Education made a failed attempt to address the demand for vacated seats and coordinate the subsequent vacancy chains via multiple reassignment rounds throughout the summer of 2014. The highly chaotic process continued well after the academic year started.[7] The overwhelming public dismay and administrative burden with this system lead the Ministry to implement a major reform through a new system the following year. Many believed that the problem with the old system had to do

[4] In particular, seats of schools available in the second stage are not wasted. Moreover, any other sequential assignment system that guarantees the existence of a straightforward SPNE respects priorities and does not waste the seats available in the second stage needs to use a wasteful mechanism in the first stage.

[5] In an earlier working paper (Andersson et al., 2020), we quantify these theoretical results using the 2015 admission data from the Swedish municipality Botkyrka. The main finding is that the waste of school seats can be dramatically reduced by organizing the two-stage admission system in such a fashion that students in the first stage are allowed to apply only to either private or public schools depending on which of the two sets of schools that is expected to have the fewest number of applicants. The empirical analysis also confirms the theoretical trade-off between incentives and wastefulness.

[6] We are indebted to an insightful anonymous referee for suggesting this result.

[7] An overall five supplementary rounds took place that year in which 13,398, 15,694, 39,037, 13,130, and 18,014 students were assigned and reassigned, respectively. See TEDMEM Report (2014).

with the organization of the timing of admissions for the two types of schools.[8] In particular, it was decided that private schools should run admissions before public schools in the new system of 2015. A second, more subtle change concerned the "rules of entry" into the second stage. Students who were admitted into a private school in the first stage were no longer permitted to apply to public schools in the second stage. However, the implementation of the new system brought forth a novel challenge related to incentives. Families found themselves under significant pressure to make a sweeping choice between private and public schools, all while remaining uncertain about the specific assignment that would result from their chosen admissions stage.[9] Therefore, families found themselves in a situation where they had to skillfully navigate the reforms and adopt a strategic approach to navigate the intricate decision-making process ahead.

The admissions systems used in many Swedish municipalities bear interesting similarities to the old and new Turkish systems. In Stockholm, for example, the public and private school admissions are entirely decentralized and independently administered with little coordination in the absence of any central oversight of an entity such as the Ministry of Education in Turkey. An epidemic problem that plagued the city's education system has been due to the fact that a student can freely enroll at several schools (possibly of the same type) before eventually deciding on her actual assignment. Much like the old Turkish system, unclaimed seats have led to severe waste and coordination problems, which has been a source of much controversy.[10] In the Swedish municipality Botkyrka, on the other hand, the legislation makes it possible for private and public schools to coordinate their admissions through the municipality. Specifically, students submit joint preferences by ranking a private school together with other public schooling options. After the private schools process the applications of those who listed them as *first* choice, the unassigned private school applicants (together with those who only listed public schools) are placed in public schools based on the remaining portion of their preferences through a central assignment. The manner in which preference submissions are constrained in Botkyrka leads to an incentive problem similar to that in the new system of Turkey: if a student's true *first* choice is not a private school, by listing only public schools, she needs to forego any admission chances to a private school; or if she still wishes to apply to a private school that is not her true *first* choice,

[8]Görmez and Coşkun (2015) conducted interviews with students, parents, and school administrators about their opinion on the admission system. For example, some administrators commented: "The system may become overwhelmed and unable to accommodate the registration process if students who plan to attend private schools are included. This often results in parents spending a considerable amount of time trying to enroll their children. Additionally, placing students who have opted for private schools and those who have not, in the same enrollment pool may create obstacles. Since students who intend to attend private schools are not making a choice from the pool anyway, it raises questions as to why the National Education is attempting to place these students."

[9]A letter from a parent to the President of the Republic of Turkey echoes this concern: "The current system forces us to make a decision between public and private schools in the beginning... Hence, many parents are forming a line before private schools fearing that they would not be admitted to any public school." (Haberturk newspaper, July 2015).

[10]This problem was, for example, discussed in a proposal submitted to the Swedish Parliament in 2015 (Motion 16:2156, 2015).

then she may miss out on a more preferred public school that would otherwise admit her. It turns out that the set of equilibria of the Botkyrka system is essentially the same as those of the new Turkish system.

1.2 Related Literature

This paper deviates from the vast majority of the existing work in the school choice literature as very few papers deal with multiple school admissions systems. An early contribution is the paper by Abdulkadiroğlu et al. (2009) that provides an informal discussion of sequential school assignments in New York City and argue that the current multi-round assignment plan may result in unstable student assignments.

Manjunath and Turhan (2016) investigate a school district where groups of schools run their admissions processes via the deferred acceptance algorithm in parallel and independently from each other. They show that the resulting school assignment is often inefficient, and offer a way to Pareto improve upon these assignments by iteratively rematching students. Turhan (2019) extends the previous model and investigates the effects of partition structures of schools on their iterated deferred acceptance mechanism. The study reveals that as school partitions get finer, students become weakly worse off, and the mechanism becomes at least as manipulable as before. Another recent experimental study, conducted by Afacan et al. (2022), explores the efficiency and stability properties of the aforementioned iterative deferred acceptance mechanism and provides substantial evidence for the effectiveness of their mechanism. Anno and Kurino (2016) also study parallel admissions that use either the deferred acceptance algorithm or the top-trading cycles mechanism, and offer new perspectives on how to operate matching markets when there are many types of markets. They find that the market-wise adaptation of strategy-proof and non-wasteful rules yields a strategy-proof rule which is Pareto undominated.

Dur and Kesten (2019) underline the conflict between efficiency and incentives in sequential systems that use either the deferred acceptance algorithm or the serial dictatorship mechanism, where students are forced to choose which stage of admissions to participate in and argue that unified admission leads to superior welfare and incentive properties. Ekmekci and Yenmez (2019) study sequential assignments that use the deferred acceptance algorithm, where the centralized admission for district schools is run separate from the individual admissions for charter schools. Their main focus is on schools' incentives to be part of a centralized system as opposed to remaining within the sequential system, and they show that such centralized admission is never incentive compatible for charter schools. Haeringer and Iehlé (2021) analyze the multi-stage college admission system in France where final assignments are gradually reached in an iterative manner based on stable mechanisms such as the deferred acceptance algorithm. Their approach, though logically different, shares conceptually similar insights with our work. Finally, Westkamp (2013) studies the German college admissions system which operates through a combination of the Boston mechanism and the college proposing deferred acceptance algorithm. He demonstrates that the set of SPNE is characterized by the set of stable matchings.[11]

Our model differs from the above cited papers in four important aspects. First, we do not assume a particular admission mechanism at the outset. All of the above cited papers base their analyses on at least one of the following mechanisms: the Boston mechanism, the deferred acceptance algorithm, the serial dictatorship mechanism, or the top-trading cycles mechanism, Second, given our point of departure that there often are institutional barriers that categorically preclude unification of admissions processes for private and public schools, the two types of admissions systems are inherently separated in our model. As described in the above, some papers in the existing literature seek to unify the admissions processes across different stages into a single centralized round. Given that no such unification is feasible in our model, this paper instead asks how the "right" sequential system should be operationalized. Third, parallel admission systems and limited sequential systems, such as the ones considered in Dur and Kesten (2019), where students can be assigned to schools in at most one stage represents special cases of our model. To our knowledge, a model with the level of generality as the one studied in this paper has not been presented before. Our general framework proves useful not only in assessing the theoretical trade-offs when the planner is unconstrained in the organization of the admission system and her choice of the underlying sets of rules, but it also contains many real-life admission systems as special cases. Finally, the sequential notion of truthfulness (straightforwardness) is new to the literature and has previously only been considered by Dur and Kesten (2019) in a less general model.

Note that this paper considers a framework where students are assigned to schools in their "own districts," so called intradistrict school choice. A practically relevant, but more demanding problem, is interdistrict school choice where students can be assigned to schools outside their own districts. This problem is investigated in a recent paper by Hafalir et al. (2022), but, as far as we are aware, no such system has been investigated in a sequential framework.

Finally, there is also a small literature that addressed sequentiality in a matching frameworks not necessarily related to school choice. One of the first papers in this literature is due to Alcalde et al. (1998), where a two-stage mechanism is proposed for a two-sided job market. Their mechanism implements the set of stable matchings in SPNE. Similarly, in many-to-one matching markets, Alcalde and Romero-Medina (2000) show that the set of stable matchings can be obtained as an equilibrium when agents play in a sequential manner.[12]

[11] In recent works, Doğan and Yenmez (2019) and Doğan and Yenmez (2018) analyze developments in the Chicago school system in which there are unified and divided enrollment systems for different types of schools. In the former paper, they particularly focus on the inefficiency of the divided enrollment system. Under substitutable choice rules, they show that students are weakly better off in the unified enrollment than the divided one. In the latter paper, they consider the welfare effects of an additional stage of assignment. By allowing students to be strategic in a two-stage game where the same set of schools are available in both stages, they compare equilibrium outcomes of one-stage and two-stage enrollment systems. Under an acylicity condition, they show that either there exists no pure strategy SPNE of two-stage system or students are weakly better off under truthful equilibrium of one-stage system. On the other hand, some students may benefit from two-stage system when acyclicity is not met.

[12] As for many-to-many matching markets, Echenique and Oviedo (2006), Romero-Medina and Triossi (2014)

The remainder of the paper is organized as follows. Section 2 introduces the school choice problem with private and public schools together with some important definitions and axioms. The sequential student assignment game is stated in Section 3. This section also includes a formal definition of the notion of straightforwardness. The general results are provided in Section 4 whereas the results related to country specific admission systems are stated in Section 5. We provide a direct mechanism equivalence result in Section 6. Section 7 concludes the paper. The appendices contain a more formal description of the extensive form student assignment game and the proofs of the results. In Appendix C, we study whether or not a unified assignment system can be sustained under our framework and provide a positive result.

2 The School Choice Problem with Private and Public Schools

This section introduces the basic ingredients of the school choice problem with private and public schools together with a number of important concepts, axioms and definitions.

2.1 The Basic Model

A school choice problem with private and public schools contains the following ingredients:

- A finite set of **schools** $S = S^{pr} \cup S^{pu}$ where S^{pr} and S^{pu} denote the set of private and public schools, respectively. Note also that each school is either a private school or a public school, i.e., $S^{pr} \cap S^{pu} = \emptyset$.
- A finite set of **students** I.
- A **capacity** vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats at school $s \in S$.
- A student **preference profile** $P = (P_i)_{i \in I}$ where P_i is the strict preference relation of student $i \in I$ over the schools in S and the option of remaining unassigned. The latter option is denoted by $s_\emptyset$.
- A school **priority ordering** $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the strict priority relation of school $s \in S$ over the students in I and the option of leaving a school seat unfilled. The latter option is denoted by $\emptyset$.

The option for a student to remain unassigned (i.e., $s_\emptyset$) is referred to as the **null-school** and it is assumed that $q_{s_\emptyset} = |I|$, i.e., that the model allows each student to remain unassigned. A school choice problem with private and public schools is denoted by $(S, I, q, P, \succ)$ and is, henceforth, referred to as a **grand problem**. For any given subset of schools $\bar{S} \subseteq S$ and any given subset of students $\bar{I} \subseteq I$, the related **subproblem** is denoted by $(\bar{S}, \bar{I}, q_{\bar{S}}, \bar{P}_{\bar{I}}|\bar{S}, \succ_{\bar{S}} |\bar{I})$.

and Sotomayor (2004), provide similar characterizations.

Here, $q_{\bar{S}} = (q_s)_{s\in\bar{S}}$ is the capacity vector of the schools in $\bar{S}$. Furthermore, $\bar{P}_{\bar{I}}|\bar{S}$ and $\succ_{\bar{S}} |\bar{I}$ are the preferences of the students in $\bar{I}$ over the schools in $\bar{S} \cup \{s_\emptyset\}$ and the priorities of the schools in $\bar{S}$ over $\bar{I} \cup \{\emptyset\}$, respectively. To simplify notation, a subproblem $(\bar{S}, \bar{I}, q_{\bar{S}}, \bar{P}_{\bar{I}}|\bar{S}, \succ_{\bar{S}} |\bar{I})$ is denoted by $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ in the remaining part of the paper.

School $s \in S$ is **acceptable for student** $i \in I$ if school s is strictly preferred to the null-school, i.e., if $sP_is_\emptyset$. Student $i \in I$ is **acceptable for school** $s \in S \cup \{s_\emptyset\}$ if student i is strictly preferred to an unfilled seat, i.e., if $i \succ_s \emptyset$. We assume that, each student $i \in I$ is acceptable for the null-school.

School priorities over subsets of students in I are assumed to be strict[13] and **responsive**.[14] This means that for any school $s \in S$, any subset of students $J \subset I$ with $|J| < q_s$, and any two students $i, j \notin J$, the following two conditions hold:

(i) $(J \cup i) \succ_s J \iff i \succ_s \emptyset$,

(ii) $(J \cup i) \succ_s (J \cup j) \iff i \succ_s j$.

These conditions mean that as long as the capacity constraint of the school is not binding, the school prefers (i) an additional acceptable student to leaving a seat empty and (ii) a student with a higher priority to a student with a lower priority when filling an additional seat.

For each student $i \in I$, let R_i denote the at least as good as relation associated with P_i, i.e.:

$$sR_is' \iff sP_is' \text{ whenever } s \neq s'.$$

Similarly, let for each school $s \in S$, $\succsim_s$ denote the at least as good as relation associated with $\succ_s$, i.e.:

$$J \succsim_s K \iff J \succ_s K \text{ whenever } J \neq K.$$

For a given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$, a **matching** is defined as a function $\mu : \bar{I} \to \bar{S} \cup \{s_\emptyset\}$ such that the number of students assigned to school $s \in \bar{S}$ does not exceed the capacity of school s, i.e., $|\mu^{-1}(s)| \leq \bar{q}_s$ for all $s \in \bar{S}$.[15][16] With slight abuse of notation, in the rest of the paper, μ_i and μ_s are used instead of $\mu(i)$ and $\mu^{-1}(s)$, respectively. Let $\overline{\mathcal{M}} \equiv \mathcal{M}(\overline{S}, \overline{I}, \overline{q}, \overline{P}, \overline{\succ})$ be the set of all possible matchings for subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$. All possible matchings under grand problem $(S, I, q, P, \succ)$ is denoted by $\mathcal{M}$, i.e., $\mathcal{M} \equiv \mathcal{M}(S, I, q, P, \succ)$.

[13] In real-life, priority classes may be thick (see the extended discussion in Abdulkadiroğlu and Andersson, 2023). If so, ties are often broken in order to obtain strict preferences, for example, by assigning each student a distinct number and break ties in accordance to those numbers (see, e.g., Abdulkadiroğlu et al., 2009). This may even by required by law for transparency reasons. In Sweden, for example, private schools rank students strictly according to rules set by the Swedish Schools Inspectorate (by Law Proposition 2009/10:165, 2009)).

[14] With slight abuse of notation, we denote the priority order of school $s \in S$ over the subsets of students with $\succ_s$.

[15] Since $q_{s_\emptyset} = |I|$, capacity constraint for the null-school never binds.

[16] Notice that, the grand problem itself can also be defined as a subproblem. As a result, matching for the grand problem is defined in the same way.

Remark 1. *The framework introduced in this section can be applied to any sequential system and not only systems that involve both private and public schools, e.g., sequential systems with only public or with only private schools (or colleges). Such systems have been considered in the earlier school choice literature as explained in Section 1.2. But even if the current paper deviates from the papers in that literature in many important aspects, there are some common modelling assumptions, for example, that all schools are capacity constrained, and have strict and responsive priorities over students (see, e.g., Dur and Kesten, 2019; Manjunath and Turhan, 2016, and also see Footnote 13). In this sense, there is no modeling assumptions that distinguishes the schools in the sets S^1 and S^2 from each other. In reality, however, private and public schools can have different objectives. Private schools may be profit maximizing and, consequently, aim to admit as many students as possible. Even if this is the case, schools are still capacity constrained in the short-run and thus need to break ties if they are oversubscribed exactly as in the considered framework. It is also not unreasonable that private schools care about their long-run reputation and one way to obtain and sustain a good reputation is to admit the best students by assigning priorities based on grades, which then makes it natural to consider responsive priorities also for private schools.* □

Next, we introduce a condition on the priority ordering $\succ$ and the capacity vector q that will be instrumental in our analysis and, in particular, in the proof of our main result (Theorem 1).

Definition 1. *Given a priority ordering $\succ$ and a capacity vector q, a* ***cycle*** *consists of distinct schools $s, s' \in S$ and students $i, j \in I$ such that the following conditions are satisfied:*

- *Cycle condition: $i \succ_s j \succ_s \emptyset$ and $j \succ_{s'} i \succ_{s'} \emptyset$.*
- *Scarcity condition: There exist (possibly empty) sets of agents $I_s, I_{s'} \subseteq I \setminus \{i, j\}$ such that $I_s \subseteq \{k \in I | k \succ_s j\}$, $I_{s'} \subseteq \{k \in I | k \succ_{s'} i\}$, $|I_s| = q_s - 1$ and $|I_{s'}| = q_{s'} - 1$.*

The priority-capacity structure $(\succ, q)$ is ***acyclic*** *if it does not have a cycle.*

Note that, under an acyclic priority-capacity structure, two schools may have different priority orderings. Ergin (2002) has defined a similar cycle condition as follows.

Definition 2. *Given a priority ordering $\succ$ and a capacity vector q, an* ***Ergin-cycle*** *consists of distinct schools $s, s' \in S$ and students $i, j, \ell \in I$ such that the following conditions are satisfied:*

- *Cycle condition: $i \succ_s \ell \succ_s j \succ_s \emptyset$ and $j \succ_{s'} i \succ_{s'} \emptyset$.*
- *Scarcity condition: There exist (possibly empty) disjoint sets of agents $I_s, I_{s'} \subseteq I \setminus \{i, j, \ell\}$ such that $I_s \subseteq \{k \in I | k \succ_s \ell\}$, $I_{s'} \subseteq \{k \in I | k \succ_{s'} i\}$, $|I_s| = q_s - 1$ and $|I_{s'}| = q_{s'} - 1$.*

In the definition of an Ergin-cycle, $\{k \in I | k \succ_s \ell\} \subseteq \{k \in I | k \succ_s j\}$. That is, if such I_s exists, then there is a set of agents $\bar{I}_s \subseteq I \setminus \{i, j\}$ such that $\bar{I}_s \subseteq \{k \in I | k \succ_s j\}$ and $\bar{I}_s = q_s - 1$. Hence, if there is an Ergin-cycle in a given problem, then there exists a cycle as defined in Definition 1. However, for some problems we may have a cycle as defined in Definition 1 but there may not exist an Ergin-cycle. Such a situation is illustrated in Example 1.

Example 1. *There are two schools $S = \{s, s'\}$ and three students $I = \{i, j, k\}$. Each school has one seat. School priorities are given as: $i \succ_s j \succ_s k \succ_s \emptyset$ and $j \succ_{s'} i \succ_{s'} k \succ_{s'} \emptyset$. In this problem, schools s and s' and students i and j constitute a cycle. However, since the first condition of Ergin-cycle is not satisfied, there does not exist an Ergin-cycle.*

Remark 2. *Ergin (2002) shows that when the priority structure is Ergin-acyclic, then the student-proposing DA mechanism always produces a Pareto efficient matching. Since our acyclicity notion is stronger, the student-proposing DA mechanism always selects a Pareto efficient matching for any problem with an acyclic structure.*[17]

2.2 Properties

Several properties shall be used to compare and evaluate matchings. For notational simplicity, we define the properties for the grand problem $(S, I, q, P, \succ)$; they directly extend to any subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$.

A matching μ is **non-wasteful** (for private schools) [for public schools] if there exists no *acceptable* student that prefers a school with an empty seat to her assigned school, i.e., if there exists no student $i \in I$ and school $s \in S$ ($s \in S^{pr}$) [$s \in S^{pu}$] such that $i \succ_s \emptyset$, $|\mu_s| < q_s$ and $sP_i\mu_i$.

A matching μ is **individually rational** if no student is assigned to a school that she finds unacceptable or she is unacceptable for. Formally, a matching μ is individually rational if $\mu_i R_i s_\emptyset$ and $i \succ_{\mu_i} \emptyset$ for all $i \in I$.

A matching μ is **fair** if whenever a student prefers some other student's assignment to her own, then the other student has a higher priority for that school than herself. Formally, μ is fair if for every $i, j \in I$, $\mu_j \in S$ and $\mu_j P_i \mu_i$ imply $j \succ_{\mu_j} i$.

A matching μ is **stable** if it is non-wasteful, individually rational and fair. We also use the following much weaker form of stability.

A matching μ is **mutually best** if there do not exist a student $i \in I$ and school $s \in S$ such that sR_is' for all $s' \in (S \cup \{s_\emptyset\})$, $i \succsim_s i'$ for all $i' \in I$ and $\mu_i \neq s$.[18]

[17] The serial dictatorship and the DA mechanisms select the same outcome whenever schools have the same priority order over the acceptable students. Since schools may have different priority orders under acyclic priority-capacity structure, such an equivalence does not immediately hold here. □

[18] This property is satisfied by most well-known mechanisms in school choice, including, e.g., the deferred acceptance algorithm and the top-trading cycles mechanism.

A student $i \in I$ prefers matching $\mu \in \mathcal{M}$ to matching $\nu \in \mathcal{M}$ if and only if she prefers μ_i to ν_i. A matching μ is Pareto dominated by matching ν if all students weakly prefer matching ν to matching μ and at least one student strictly prefers matching ν to matching μ, i.e., matching μ is **Pareto dominated** by matching matching ν if $\nu_i R_i \mu_i$ for all $i \in I$ and there exists at least one student $j \in I$ where $\nu_i P_i \mu_i$. A matching μ is **Pareto efficient** if there does not exist another matching ν which Pareto dominates μ.

A **mechanism** ϕ is a systematic procedure that selects a matching for any given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$. The outcome selected by mechanism ϕ in subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ is denoted by $\phi(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$, and the match of student $i \in \bar{I}$ and school $s \in \bar{S}$ are denoted by $\phi_i(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$ and $\phi_s(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$, respectively.

A mechanism ϕ is non-wasteful <individually rational> [fair]{stable} (Pareto efficient) <mutually best> if it selects a non-wasteful <individually rational> [fair] {stable} (Pareto efficient) <mutually best> matching for any given subproblem $(\bar{S}, \bar{I}, \bar{q}, \bar{P}, \bar{\succ})$.

3 The Sequential Student Assignment Game

The essential difference between the school choice model presented in the previous section and the standard model (Abdulkadiroğlu and Sönmez, 2003) is that the set of schools is partitioned into a set of private schools and a set of public schools, i.e., $S = S^{pr} \cup S^{pu}$ and $S^{pr} \cap S^{pu} = \emptyset$. This seemingly small deviation from the standard model makes a large difference when coupled with a sequential admission system. As will be exemplified in Section 5, many countries separate admission to private schools from admission to public schools. Students may, for instance, first have the option to apply to private schools and the private school admission is then followed by a centralized public school admission (e.g., the admission system used in Turkey between 2015–2017). Such two-stage admission processes also introduce many details that need to be analyzed. For example, if the admission to private schools proceeds the admission to public schools, can a student that is assigned to a private school also participate in the admission to public schools? If so, can the student "keep" her placement at the private school or does she have to "give up" the placement to participate in the public school admission?

To analyze the sequential school choice model under different sets of rules, a game in which play proceeds in a sequence of three rounds is considered. The analysis is restricted to the case where $\bar{S} \in \{S^{pr}, S^{pu}\}$, i.e., the case where the admission to private and public schools can be separated from each other in two different rounds of the game (as in, e.g., the above-mentioned Turkish system). Because the considered admission systems, the game, and the theoretical findings can be explained without introducing too much notation, a simpler exposition is adopted in the main body of the paper for ease of exposition. The interested reader is referred to Appendix A for the formal technical definitions.

The sequential student assignment game contains three separate rounds. In Round 1, only private or only public schools are available. The set of available schools in Round 1 is denoted by S^1 and, consequently, $S^1 \in \{S^{pr}, S^{pu}\}$. Students that can participate in Round 1 are collected in the set I^1. Without loss of generality, we set $I^1 = I$ in our analysis, i.e., any student is allowed to participate in Round 1. The students in I^1 submit a rank order list over the schools in $S^1 \cup \{s_\emptyset\}$. If only private schools (public schools) are available in Round 1, then only public schools (private schools) are available in Round 2. The set of participating schools and students in Round 2 are denoted by S^2 and I^2, respectively. The students in I^2 submit a rank order list over the schools in $S^2 \cup \{s_\emptyset\}$. Because we only analyze systems in which admission to private schools is separated from admission to public schools, it follows that $S^2 = S \setminus S^1$ and, consequently, that $S^1 \cup S^2 = S$ and $S^1 \cap S^2 = \emptyset$. In Round 3, the students are assigned to schools. However, whether or not a student can participate in Round 2 and what alternatives the students have in Round 3 depend on the rules of the game.

The **rules of the game** are given by the mechanisms ϕ^1 and ϕ^2 that are used for assigning students to schools in Rounds 1 and 2, respectively, together with two correspondences, ψ and γ. Here, the correspondence ψ determines the set of students that are allowed to participate in Round 2, i.e., ψ determines I^2, and the correspondence γ provides the alternatives for the students in Round 3. The correspondence ψ may or may not depend on the outcome of the mechanism ϕ^1. The analysis in this paper is restricted to two cases that both describe real-life admission systems, namely when the correspondence ψ prescribes that:[19]

- only students that are assigned to the null-school in Round 1 are allowed to participate in Round 2, i.e., $I^2 = \{i \in I : \phi^1_i(S^1, I^1, q^1, P^1, \succ^1) = s_\emptyset\}$,
- all students in I are allowed to participate in Round 2, i.e., $I^2 = I$.

The correspondence γ provides the alternatives for the students in Round 3. These alternatives do not only depend on the outcomes of the mechanisms ϕ^1 and ϕ^2, but also on if the student is allowed to participate in Round 2 or not. More precisely, the correspondence γ includes the following situations:

(1) If student i is not allowed to participate in Round 2, then she has to select the Round 1 assignment.

(2) If student i is allowed to participate in Round 2, then the selection can take three different forms:

    (a) student i can select the Round 1 or the Round 2 assignment if she participates in Round 2,

[19]Note that it is easy to extend the correspondence ψ to also include the intermediate cases in which some of the assigned students in Round 1 can participate in Round 2 and some of the unassigned students in Round 1 cannot participate in Round 2. All results presented in this paper still hold for these intermediate cases.

(b) student i has to select the Round 2 assignment (the Round 1 assignment) if she (does not) participates in Round 2,

(c) student i has to select the Round 1 assignment (the Round 2 assignment) if she participates in Round 2 and the Round 2 assignment is the null-school (distinct from the null-school).

Note that independently of if case (1) or (2) prevails, the correspondence γ is a subset of assignments from Rounds 1 and 2. Note also that even if the rules of the game are generally defined in the above, many existing sequential school admission systems fit in this framework. This will be discussed and exemplified in Section 5. The sequential admission process and its relation to the rules of the game can formally be described as:

- *Round* 1. Each student $i \in I^1$ submits a rank order list P_i^1 over the schools in $S^1 \cup \{s_\emptyset\}$. The mechanism ϕ^1 assigns each student $i \in I^1$ to school $\phi_i^1(S^1, I^1, q^1, P^1, \succ^1)$ in $S^1 \cup \{s_\emptyset\}$.
- *Round* 2. The correspondence ψ determines the set of students that are allowed to participate in Round 2. Each student that is allowed to participate in Round 2, i.e., the students in the set $i \in I^2$, submit a rank order list P_i^2 over the schools in $S^2 \cup \{s_\emptyset\}$. The mechanism ϕ^2 assigns each participating student $i \in I^2$ to school $\phi_i^2(S^2, I^2, q^2, P^2, \succ^2)$ in $S^2 \cup \{s_\emptyset\}$.
- *Round* 3. The correspondence γ provides the school alternatives for each student $i \in I$ based on the assignments in Rounds 1 and/or 2, and each student $i \in I$ selects one of the alternatives prescribed by γ.

Under individual rationality, one can think of students that are allowed to but do not wish to participate in Round $j \in \{1, 2\}$ as students, who simply submit the null-school as their top choice when they report their preferences. In such a case, students can always (and without loss of generality) be allowed to keep their Round 1 assignments by submitting $s_\emptyset$ as their top choice in Round 2. To simplify our analysis, a student who is allowed to but does not wish to participate in Round $j \in \{1, 2\}$ ranks $s_\emptyset$ as her top choice. That is, I_1 consists of all students, and I_2 consists of all students that are allowed to participate in Round 2.

Throughout the paper, it is assumed that the moves in Round 1 are observed before Round 2 begins, and that the moves in Round 2 are observed before Round 3 begins. Participating students also move simultaneously within each round.

3.1 The Extensive Form Game

The above sequential process can be modeled as an **extensive form game**. A detailed description of this game is provided in Appendix A. The informal definition follows next.

The game proceeds in three rounds. The **initial node** of the game, say h_1, can be thought of as Round 1, and it is based on subproblem $(S^1, I^1, q^1, P^1, \succ^1)$. In this node, a **subgame** is played.[20] In this subgame, each student or, equivalently, each **player**, $i \in I^1$, reports a ranking P_i^1 over the schools in $S^1 \cup \{s_\emptyset\}$. The reported ranking belongs to the set of all possible (strict) rankings over the schools in $S^1 \cup \{s_\emptyset\}$ and it need not be truthful. The reported ranking P_i^1 also represents the **action** of student $i \in I^1$. The actions played by the students in I^1 at node h_1 together with the rules of the game will take the extensive form game to an **intermediate node**, say h. Note that node h belongs to some set of possible nodes H^2 but exactly what node in H^2 that the game ends up in depends on the actions as well as the rules of the game. Node h is based on subproblem $(S^2, I^2, q^2, P^2, \succ^2)$ and it can be thought of as Round 2. A subgame, where each student $i \in I^2$ reports her ranking P_i^2 over the schools in $S^2 \cup \{s_\emptyset\}$, is then played at node h. Note here that the subgame played at node h is defined based on the play at the initial node h_1, i.e., the subgame at node h depends not only on the rules of the game but also on the **history** of the game. In a similar manner as in the above, the play at node h together with the rules of the game will take the extensive form game to a **penultimate node**, say $\bar{h}$. Node $\bar{h}$ can be regarded as Round 3 where each student selects a school (possibly the null-school) from the alternatives proposed to her. Again, the choices at node $\bar{h}$ depend on the history and the rules of the game. In this sense, the initial node h_1 and the strategies played in the various subgames define a unique penultimate node of the extensive form game.

Because the considered game is a sequential game, we adopt the notion of a **Subgame Perfect Nash Equilibrium** (SPNE, henceforth) as the equilibrium concept. Such an equilibrium is a strategy profile that induces a **Nash Equilibrium** in every (proper) subgame.

3.2 The Notion of Straightforwardness

It is well established in the school choice literature that students often can gain by misrepresenting their preferences over schools and that they sometimes use this possibility to their advantage (see, e.g., Abdulkadiroğlu et al., 2005; Dur et al., 2018; Pathak and Sönmez, 2008, 2013). Obviously, students can have different strategies also in the considered sequential game and not all of these strategies involve truthful reports over the available schools in each round. Because truth-telling plays an important role in the school choice literature, we also need to introduce such a concept for the considered extensive form game. This is the notion of straightforwardness.

Definition 3. *Given rules of the game $(\phi^1, \phi^2, \psi, \gamma)$, a student $i \in I$ plays a **straightforward** strategy if it involves the following actions:*

- Round 1*: Student i reports her true ranking over the schools in $S^1 \cup \{s_\emptyset\}$.*
- Round 2*: Following each node at H^2, student $i \in I^2$ reports her true ranking over the schools in S^2 that are acceptable and strictly more preferred to her Round 1 assignment that she is allowed to hold (note that the Round 1 assignment might be $s_\emptyset$) followed by $s_\emptyset$, i.e., any other school is ranked below $s_\emptyset$.*

[20]Note that this does not have to be a proper subgame.

- Round 3*: Following each node at H^3, student i selects her most preferred school out of the schools prescribed by γ.*

Each action satisfying the conditions above following the corresponding node is called a straightforward action. The notion of straightforwardness does not only capture the idea that students report truthful rankings over the schools, but the notion also captures rationality in the sense that any student that plays a straightforward strategy at the same time makes sure that their current situation in the game weakly improves according to her true preferences. For example, before Round 1 starts, no student is assigned to any school and, therefore, each student that plays a straightforward strategy reports a ranking over the schools that are (weakly) preferred to her current situation of being unassigned. Similarly, any student that participates in Round 2 and plays a straightforward strategy will weakly improve her current situation. For example, if the student is assigned the Round 2 assignment whenever it is distinct from the null-school, the student cannot lose anything by adopting the action to only report a ranking over the schools in $S^2 \cup \{s_\emptyset\}$ such that all ranked options except $s_\emptyset$ are strictly preferred to her Round 1 assignment. The same holds for Round 3 where it, obviously, is rational to select the most preferred school of the available alternatives.

From a normative point of view, the notion of straightforwardness is weaker than the well-known notion of strategy-proofness for one-round assignment (defined in the usual sense, see, e.g., Abdulkadiroğlu and Sönmez, 2003). This is because strategy-proofness also means that truthful-telling is a weakly dominant strategy. Moreover, strategy-proof mechanisms are "fair" because strategy-proofness diminishes the harm to people who do not strategize or do not strategize well (see, e.g., Subsection 7.1 in Abdulkadiroğlu et al., 2006). On the other hand, requiring the truthful-telling strategy profile to be a dominant strategy equilibrium might be strong in practice. For instance, some experimental studies find that some non-strategy-proof mechanisms produce higher truth-telling rates than strategy-proof mechanisms (e.g., Cerrone et al., 2022). Thus, for the mechanism designer, straightforwardness might not be a demanding property, especially if the mechanism designer knows that there is a trade-off between straightforwardness and non-wastefulness as will be shown shortly.

Finally, a Subgame Perfect Nash Equilibrium (SPNE) is straightforward if each student in I plays a strategy composed of only straightforward actions.

4 General Sequential School Admission Systems

Before investigating the specific sequential systems used in different countries, we first provide a comprehensive analysis of a canonical sequential school admission game that allows us to understand the complex interplay between the round-specific mechanisms and the correspondences that govern the rules of the game. As will be demonstrated in this section, both the mechanisms and the rules of the game need to satisfy very specific properties to guarantee the existence of a straightforward SPNE. Specifically, we show that there is a trade-off between the existence of a straightforward SPNE and non-wastefulness.

We progressively identify the conditions that are essential for a possibility result. To do so, at the outset we provide two sets of impossibility results. These negative results will pin down the exact conditions that are needed for the mechanisms and the rules of the game to guarantee the existence of a straightforward SPNE that induces an outcome satisfying the desired properties. First, we focus on the impossibility results driven specifically by the properties of the mechanisms ϕ^1 and ϕ^2.

Proposition 1. *For any $i, j \in \{1, 2\}$ and $i \neq j$, if the mechanism ϕ^i is individually irrational <wasteful> [unfair], then for any (ψ, γ, ϕ^j) there exists a problem $(I, S, q, \succ, P)$ such that either none of the SPNE are straightforward or any straightforward SPNE induces an individually irrational <wasteful> [unfair] matching.*

Proposition 1 implies that for a positive result one must inevitably restrict attention to mechanisms ϕ^1 and ϕ^2 that are individually rational, non-wasteful and fair. This is also natural since in school choice contexts, assigning students to unacceptable schools, violating the priorities (obtained, e.g., from a centralized exam or predetermined criteria) or wasting some school seats are often deemed highly undesirable.

In the following, we also consider mechanisms that satisfy an auxiliary property of mutually best. It is easy to verify that fairness and non-wastefulness imply mutual best. Given the restrictions necessitated by Proposition 1, we next provide a second set of impossibility results driven by the interaction between correspondences ψ and γ.[21]

Proposition 2. *Let $(\phi^1, \phi^2, \psi, \gamma)$ be the rules of the game. Then:*

1. *If ϕ^1 is non-wasteful and individually rational, ϕ^2 is non-wasteful, and (ψ, γ) prescribe that only students that are assigned the null-school in Round 1 can participate in Round 2, then there exists a problem $(I, S, q, \succ, P)$ which admits a unique SPNE that is not straightforward under the induced game.*

2. *If ϕ^1 is non-wasteful, mutually best, and individually rational, ϕ^2 is mutually best, and (ψ, γ) prescribe that all students are allowed to participate in Round 2 but have to select their Round 2 assignments when they actively participate in Round 2, i.e., rank some school over $s_\emptyset$, then there exists a problem $(I, S, q, \succ, P)$ which admits no straightforward SPNE under the induced game.*

3. *If ϕ^1 is mutually best, ϕ^2 is non-wasteful and individually rational, and (ψ, γ) prescribe that all students are allowed to participate in Round 2, then there exists a problem $(I, S, q, \succ$*

[21] We would like to emphasize that in the proofs of Propositions 1 and 2 we use examples in which priority orders satisfy our acyclicity condition.

$, P)$ *which admits a unique straightforward SPNE whose outcome is wasteful for the schools in* S^1 *under the induced game.*

4. *If* ϕ^1 *is non-wasteful and individually rational,* ϕ^2 *is mutually best, and* (ψ, γ) *prescribe that all students are allowed to participate in Round 2 and can select their most preferred outcomes from Rounds 1 and 2, then there exists a problem* $(I, S, q, \succ, P)$ *which admits a straightforward SPNE outcome that is wasteful for the schools in* S^2 *under the induced game.*

Table 1: *Summary of the results in Proposition 2.* ϕ^1 *and* ϕ^2 *are individually rational, non-wasteful, and mutually best.*

		γ		
		Available Choices		
		Round 1 and Round 2	Only Round 2 (Round 1) if (not) participate Round 2	Round 1 if unassigned in Round 2 otherwise Round 2
ψ	Only unassigned can participate Round 2	no straightforward SPNE	no straightforward SPNE	no straightforward SPNE
	All can participate Round 2	wasteful for S^1 and S^2	no straightforward SPNE wasteful for S^1	wasteful for S^1

Given that ϕ^1 and ϕ^2 are individually rational, non-wasteful and satisfy mutually best, the first two parts of Proposition 2 suggest that unless all agents are allowed to participate in Round 2 or forced to give up their Round 1 assignments as a precondition to participate in Round 2, the existence of a straightforward SPNE cannot be guaranteed. Moreover, the third part of Proposition 2 suggests that if all agents are allowed to participate in Round 2, then some problems always lead to wasteful SPNE outcomes for the schools in S^1. Finally, the last part of Proposition 2 says that if all students are allowed to participate in Round 2 without giving up their Round 1 assignments, then some problems inevitably lead to wasteful SPNE outcomes for the schools in S^2. The results from Proposition 2 are summarized in Table 1.

The findings in Proposition 2 establish how the correspondences ψ and γ must be chosen in order to guarantee the existence of a straightforward SPNE for any problem. More precisely, the first two parts of Proposition 2 imply that all students should be allowed to participate in Round 2 without giving up their Round 1 assignment. However, if such rules are adopted, for some problems, a straightforward SPNE induces an equilibrium outcome in which some school seats in Round 1 are wasted. Furthermore, if students are allowed to participate in Round 2 without giving up their Round 1 assignment, for some problems, a straightforward SPNE induces an equilibrium outcome in which some school seats in Round 2 are wasted. These findings highlight the trade-off between straightforwardness and non-wastefulness.

Now consider the correspondences ψ^* and γ^* where the former prescribes that all students are allowed to participate in Round 2, and the latter prescribes that any student that is assigned a school distinct from the null-school in Round 2 has to select her Round 2 assignment.[22] Let both mechanisms ϕ^{1*} and ϕ^{2*} be the unique strategy-proof, individually rational, non-wasteful and fair mechanism, i.e., the student-proposing deferred acceptance (DA) mechanism.[23] Given the above insights, the rules of the game $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$ are not only well-defined but they are essentially the only remaining set of options that can potentially guarantee the existence of straightforward SPNE while reducing waste and respecting priorities. As illustrated in the following theorem, this only remaining set of options will, in fact, do the job provided that the priority-capacity structure is acyclic.

Theorem 1. *Suppose that the rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$. Then, for any sequential problem with acyclic priority-capacity structure, the induced game has a straightforward SPNE, and the seats for the schools in S^2 are never wasted at any equilibrium outcome.*

Theorem 1 uncovers an additional requirement for a possibility result. Namely, the priority-capacity structure should satisfy our strong acyclicity notion. Three additional remarks are in order. First, if ϕ^{1*} and ϕ^{2*} are given by the DA, a straightforward SPNE outcome of $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$ is fair and individually rational since these properties are possessed by the mechanisms used in Rounds 1 and 2. If the goal is to guarantee the existence of straightforward SPNE with an outcome which is fair, individually rational, and non-wasteful for the schools in Round 2, then the mechanism to use in Round 2 must be the DA.[24] It is worth mentioning that the necessity of using DA is not implied solely by fairness. In fact, if guaranteeing existence of straightforward SPNE is dropped out from our goal, then using other fair mechanisms may fulfill other remaining desired properties.

Second, when $(\succ, q)$ is acyclic and ϕ^{1*} and ϕ^{2*} are given by the DA, then under $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$ it is a weakly dominant strategy for all students to play a straightforward strategy. This follows from the acyclicity of the priority-capacity structure, i.e., that no student can affect the assignment of some other student in Round 1 who can affect her assignment at some school in Round 2.

Third, Theorem 1 holds whenever $(\succ, q)$ is acyclic. As demonstrated in the following proposition, this assumption is crucial for the result to remain true. More precisely, if $(\succ, q)$ has a cycle, then the existence of a straightforward SPNE is not guaranteed even if the rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$. This is more precisely stated in the following result.

[22] In other words, if she is assigned to some school in Round 2, then only that school will be available in Round 3.

[23] Since any other individually rational, non-wasteful and fair mechanism is not strategy-proof, one can find a problem such that there does not exist a straightforward SPNE in the associated game with rules ψ^* and γ^*, and stable mechanisms other than DA. In fact, this observation can be generalized to any mechanism that is not strategy-proof.

[24] However, it is still possible to use a limited (Pareto inferior) version of DA that preserves strategy-proofness in Round 1. For example, a version of DA where some seats are first removed before applying the mechanism to the remaining problem would work. Our proof for Theorem 1 can be adapted also for this version.

Proposition 3. *Suppose that rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$. Then, there exists a cyclical $(\succ, q)$ such that the induced game does not have a straightforward SPNE.*

Proposition 3 and Theorem 1 together characterize the conditions for $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$ to have a straightforward SPNE.[25] Even if Theorem 1 is positive in the sense that it helps us to understand under which conditions a straightforward SPNE exists, Proposition 3 warns that cyclic priority and quota profiles may transform a possibility into into an impossibility. More precisely, since acyclicity is a restrictive condition, Theorem 1 and Proposition 3 effectively advises practitioners, who would like design sequential admission systems satisfying the axioms considered in this paper, to use essentially identical admission criteria at all schools, e.g., identical priority orders. This may become problematic if it is not possible to regulate the admission criteria for all schools in S^1 and S^2, say the private schools. But even if there are sequential systems where the priority and quota profile not necessarily are acyclic (see Section 5.3 for an example), it is not uncommon with acyclic priority-capacity structure in other types of admission systems. For example, priorities are based exclusively on central exams in, e.g., college admissions in China (Chen and Kesten, 2017) and high school admissions in Taiwan (Dur et al., 2020). The network "Matching in Practice"[26] reports that admissions in some secondary schools in Europe are based on acyclic criteria such as academic performance (e.g., Finland) and centralized test scores (e.g., Hungary). Such admission criteria are more common in higher education. So, even if acyclicity is a restrictive condition, it is used in non-sequential systems and it is, consequently, not unlikely that policy makers implement it also in systems with sequential admissions.

5 Sequential School Admission Systems in Europe

This section describes three different sequential school admission systems and explains how they can be interpreted in the considered sequential school admission framework. Note that these descriptions does not capture all details, timing issues, or practical aspects of the actual real-life systems. The considered model is, however, sufficiently close to these systems to derive conclusions that we believe can shed important insights on real-life sequential admission systems. Importantly, in relation to the investigated school choice systems, the model analyzed in this paper should be regarded as minimalist, in the sense of Sönmez (2023), meaning that it is a reasonable, but not a perfect, description of reality.

[25] In the proof of Proposition 3, the relationship between acyclicity and Ergin-acyclicity implies that Ergin-acyclicity does not guarantee the existence of a straightforward SPNE when the rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$.

[26] See `https://www.matching-in-practice.eu`.

5.1 The Old Turkish System

The Ministry of Education (MOE) in Turkey implemented some reforms in the Turkish high school admissions in 2014. In this system, students participated in two nationwide high school entrance exams in the eighth grade. A Score for Placement (SFP) was calculated for each student by taking a weighted average of the test scores from the two nationwide entrance exams and the Grade Point Average (GPA). All public school admissions are administered by MOE via a centralized clearinghouse, and priorities of public schools over students were based on the SFPs. Accordingly, all public schools had the same priorities over the students. However, private schools were not included in the centralized match as MOE lacks jurisdiction over private school admissions. Note, however, that private schools often had different sets of acceptable students in the sense that they had different views on the lowest acceptable SFPs (the so-called cut-off scores).[27] It is easy to see that the priority-capacity structure is acyclic.

The old Turkish system worked as follows. In Round 1, each student submitted a ranking over public schools. Students were then assigned to public schools based on the serial dictatorship mechanism where the priorities were given by the SFPs. In Round 2, all students were allowed to participate independently of the outcome in Round 1. Participating students applied to private schools and were admitted to private schools in a decentralized way. In Round 3, the students selected their most preferred outcome from Rounds 1 and 2. Formally, this means that the correspondence ψ prescribes that $I^2 = I$ and that the correspondence γ is described by the case (2a) from Section 3.[28] For a more detailed description of the old Turkish system, see Application Manual of Ministry of Education of Turkey (2014).

It is next demonstrated that any SPNE outcome of the old Turkish system is fair and individually rational but it may fail to be non-wasteful. The latter conclusion is demonstrated by means of an example and it intuitively holds because students need not give up their Round 1 assignments to participate in Round 2. Consequently, if some student that was assigned a school in Round 1 selects her Round 2 assignment, a school seat in Round 1 may be wasted.

Proposition 4. *Any SPNE outcome of the old Turkish system is individually rational and fair.*[29]

Example 2. *To demonstrate that the old Turkish system is wasteful, let* $S^1 = S^{pu} = \{s_1\}$*,* $S^2 = S^{pr} = \{s_2\}$*, and* $I = \{i_1, i_2, i_3\}$*. Further, suppose that each school has two available seats, and that student preferences are given by:* $s_2 P_{i_1} s_1 P_{i_1} s_\emptyset$*,* $s_1 P_{i_2} s_2 P_{i_2} s_\emptyset$ *and* $s_1 P_{i_3} s_2 P_{i_3} s_\emptyset$*. The school priorities are given by* $i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset$ *for all* $s \in S$*. In this example, a strategy*

[27]Most private schools also form their priorities via SFPs. There are only a few private schools (mostly international schools) that use different weights.

[28]In an earlier working paper (Andersson et al., 2020), we demonstrate that the decentralized admission game for private schools has a unique SPNE outcome which is equivalent to the outcome of the (constrained) serial dictatorship mechanism under true preferences and true cut-off scores.

[29]Although in the old Turkish system, schools have the same ranking over the students, this result is also true under acyclic priority and quota profiles.

where each student submits her true preferences over the available schools in any node in Rounds 1 and 2, and then selects her most preferred school in any node in Round 3 is an SPNE. Given such strategy, students i_1 and i_2 are admitted to school s_1 in Round 1, and students i_1 and i_2 are admitted to school s_2 in Round 2. Consequently, the outcome in Round 3 is that student i_1 is assigned to school s_2, student i_2 is assigned to school s_1, and student i_3 is assigned to the null-school $s_\emptyset$. Hence, one seat at school s_1 and one seat at school s_2 is wasted. □

Finally, we note that Proposition 4 and Example 2 hold also when private school assignment is done before the public school assignment as long as the correspondences ψ and γ are defined as in the old Turkish system. Hence, the wastefulness in the assignment cannot be mitigated by just changing the order of assignment.

Because of the option of holding offers in both rounds and unrestricted entrance to Round 2, one can see similarities between the old Turkish system and parallel assignment systems in which students can apply to public and private schools at the same time and then pick the best offer she receives. In this sense, the old Turkish system and the parallel assignment system adopted by most Swedish municipalities have common equilibrium properties.

5.2 The New Turkish System

As demonstrated in the previous example, the old system resulted in a large number of vacant seats in public schools. This led to many rounds of reassignments which hardly resolved the problem. To mitigate these issues, MOE replaced the admissions system in 2015. The main difference compared to the old system was that the roles of the private and public schools were reversed, and that the rules related to which students that are allowed to participate in Round 2 (i.e., the correspondence ψ) was changed. More precisely, in Round 1, students applied to private schools and the admission was, again, organized in a decentralized fashion. Only students that remained unassigned after Round 1 were allowed to participate in Round 2. The participating students submitted a ranking over public schools. Formally, this means that the correspondence ψ prescribes that $I^2 = \{i \in I : \phi_i^1(S^1, I^1, q^1, P^1, \succ^1) = s_\emptyset\}$ and that correspondence γ is described by case (1) from Section 3. Note also that this procedure will assign each student to at most one school in Round 3. As demonstrated next, in contrast to the old Turkish system, any SPNE in the new Turkish system is stable. For a more detailed description of the new Turkish system, see Application Manual of Ministry of Education of Turkey (2015).

Proposition 5. *Any SPNE outcome of the new Turkish system is stable, i.e., non-wasteful, individually rational, and fair.*[30,31]

As in the old Turkish system, the decentralized admission game for private schools has a unique SPNE outcome that is equivalent to the outcome of the (constrained) serial dictatorship mecha-

[30]Since all schools rank the acceptable students in the same order, there is a unique stable matching.

[31]As in Proposition 4, this result is also true under acyclic priority-capacity structures.

nism under true preferences and true cut-off scores (see Appendix D in Andersson et al., 2020) and the result holds also when reversing the roles of the public and private schools in the assignment process.

Although any SPNE outcome of the new Turkish system is stable, Proposition 2 implies that for some problems under the new Turkish system there does not exist a straightforward SPNE. To be more specific, in the new system, students are facing a risky decision: either register for the best available private school and do not apply in the second round, or give up your private school assignment and try your chances for a better public school (this concern was also ventilated in a letter from a parent to the President of Republic of Turkey, see Footnote 9). Another interesting implication of Propositions 1, 2 and 5 is the following corollary.

Corollary 1. *Suppose that all schools in* S *have the same relative priorities over acceptable students. Then, the new Turkish system's rules are the unique rules of the game such that any SPNE outcome is stable.*

Finally, when we compare the equilibrium outcomes under the old and new Turkish systems, Propositions 4 and 5 imply the following corollary.

Corollary 2. *Suppose that all schools in* S *have the same relative priorities over acceptable students. Then, the unique equilibrium outcome of the new Turkish system (weakly) Pareto dominates any equilibrium outcome of the old Turkish system.*

5.3 The Botkyrka System

In Sweden, the municipalities determine the admission process for public schools even if they are guided by national legislation (*Skollagen*). Admission to public schools is centralized within each municipality and the priorities for public schools are decided based on relative distance to schools. The latter means that public schools typically have different priorities over students. Admission to private schools is decentralized but is also guided by legislation. The municipality of Botkyrka (south-west of Stockholm) will be used to illustrate the Swedish system.[32]

In the Botkyrka system, parents rank schools in a centralized online application system containing all private and public schools in the municipality. The system allows parents to rank at most three schools in total. However, an application to a private school will only be considered if it is top-ranked (i.e., the most preferred school according the students submitted ranking). In Round 1, the application for each student that ranks a private school as a top-choice is forwarded to that private school. All private schools have a waiting list, and priority is given on first-come-first-served basis. The private schools then match the applications to their waiting lists, and returns the names of the students they wish to admit to the centralized application system. In Round 2, only students that remain unassigned after Round 1 are allowed to participate. The

[32]See Andersson (2017) for a detailed description of the Swedish school choice system, and Kessel and Olme (2018) for a more detailed description of the Botkyrka system.

ranking of public schools for these students are forwarded to a centralized clearinghouse and the admission is done via the (student proposing) deferred acceptance algorithm. This procedure will assign each student to at most one school in Round 3.[33]

Note that the correspondences ψ and γ are defined as in the new Turkish system but the mechanisms ϕ^1 and ϕ^2, as well as the priority orders, differ from the new Turkish system. This will also have the consequence that the properties of any SPNE outcome differs between the new Turkish system and the Botkyrka system. More precisely, an SPNE outcome need not be non-wasteful for public schools in the Botkyrka system unless there is a SPNE where students play straightforward actions in Round 2.[34] Furthermore, an SPNE outcome of Botkyrka system need not be fair. The latter result is illustrated by means of an example.

Proposition 6. *Any SPNE outcome of Botkyrka system is non-wasteful, fair for private schools, and individually rational. Moreover, when students report true preferences in Round 2, any SPNE outcome is non-wasteful for the public schools.*

Example 3. *Let $S^1 = S^{pr} = \{s_1\}$, $S^2 = S^{pu} = \{s_2, s_3\}$, and $I = \{i_1, i_2, i_3\}$. Suppose further that each school has one available seat, and that student preferences are given by: $s_2 P_{i_1} s_1 P_{i_1} s_\emptyset$, $s_2 P_{i_2} s_3 P_{i_2} s_\emptyset$, and $s_3 P_{i_3} s_2 P_{i_3} s_\emptyset$. The school priorities are given by $i_1 \succ_{s_1} i_2 \succ_{s_1} i_3 \succ_{s_1} \emptyset$, $i_3 \succ_{s_2} i_1 \succ_{s_2} i_2 \succ_{s_2} \emptyset$, and $i_2 \succ_{s_3} i_3 \succ_{s_3} i_1 \succ_{s_3} \emptyset$. In this example, straightforward play by all students is an SPNE. The outcome of this strategy is that student i_1 is assigned to school s_1, student i_2 is assigned to school s_2, and student i_3 is assigned to school s_3. But this outcome is not fair since student i_1's priority is not respected at public school s_2 since s_2 is assigned to student i_2 but $s_2 P_{i_1} s_1$ and $i_1 \succ_{s_2} i_2$.* □

Proposition 6 states that any SPNE outcome is non-wasteful for the public schools when students submit their true preferences in Round 2 (this is a weakly dominant strategy in each subgame in that round). In the following example, it is illustrated that one may end up with wasteful equilibrium outcomes for the public schools when students play other equilibrium strategies.

Example 4. *Let $S^1 = S^{pr} = \{s_1\}$, $S^2 = S^{pu} = \{s_2, s_3\}$, and $I = \{i_1, i_2\}$. Further, suppose that each school has one available seat, and that student preferences are given by: $s_2 P_{i_1} s_1 P_{i_1} s_3 P_{i_1} s_\emptyset$ and $s_3 P_{i_2} s_2 P_{i_2} s_\emptyset$. The school priorities are given by $i_1 \succ_{s_1} i_2 \succ_{s_1} \emptyset$, $i_2 \succ_{s_2} i_1 \succ_{s_2} \emptyset$, and $i_1 \succ_{s_3} i_2 \succ_{s_3} \emptyset$. In this example, the following strategy profile is an SPNE: (a) both students play their straightforward action in Round 1, (b) both students rank the school in which they have the highest priority in Round 2 at the top whenever the other student also participates in Round 2, and rank their most preferred school in Round 2 at the top otherwise. In the outcome related to this SPNE, student i_1 is assigned to school s_1 and student i_2 is assigned to school s_3. Hence, the seat at s_2 is wasted.* □

[33] In order to be consistent with our model, we take application to private and public schools done sequentially.

[34] If we focus on the simultaneous preference revelation game in which students submit a single preference list, any Nash equilibrium outcome is non-wasteful, individually rational, and fair for private schools.

Although, we may end up with wasteful and/or unfair equilibrium outcomes under Botkyrka system, both Botkyrka and the new Turkish systems are structurally identical. In particular, if we consider Botkyrka system under homogeneous priorities for all schools, then any equilibrium outcome will be stable (see the proof of Proposition 5).

6 Direct Mechanism Approach: A Revelation Principle

This section asks the following question: what direct mechanism is implemented by our two-stage indirect mechanism?[35] The answer can be useful in assessing how close we can get to implementing the outcome of a one-shot DA mechanism that would have been ideal in the absence of the institutional restrictions one faces in the context considered in this paper. This section provides a direct mechanism whose outcome is equivalent to the straightforward equilibrium outcome under the rules stated in Theorem 1 when the priority-capacity structure is acyclic. That is, the DA mechanism is run twice by considering the set of schools available in Rounds 1 and 2. However, different from the sequential game, students report their preferences over *all* schools before the DA mechanism is run to determine the outcome. This mechanism is called the **Decomposing DA Mechanism** and it can formally be describes as follows:

- *Step 0:* Each student submits her preferences over all schools, including the null-school $s_\emptyset$.
- *Step 1:* Apply DA by considering each student's preferences restricted to schools in $S^1 \cup \{s_\emptyset\}$. Let μ^1 be the outcome of in this step.
- *Step 2:* Apply DA by considering each student's preferences restricted to those schools in S^2 that she weakly prefers to her assignment under μ^1 and $\{s_\emptyset\}$. Let μ^2 be the outcome of this step.
- *Step 3:* Each student i is assigned to the more preferred school among μ^1_i and μ^2_i.

Notice that, the above mechanism selects a matching for each problem. Since DA is adopted in both Steps 1 and 2, the resulting matching in Step 3 is fair and individually rational.[36] Moreover, since in Step 2 students only apply to the schools they prefer to their assignment in Step 1, seats at schools in S^2 are never wasted. That is, the decomposing-DA mechanism caries over some of the desirable properties of the DA mechanism (see Theorem 2). However, as illustrated in the next example, the decomposing DA mechanism is not strategy-proof (when priority-capacity structure is cyclic) and some seats in S^1 might be wasted.

[35] We thank an insightful referee for raising this important question whose suggestion motivated this section.

[36] This follows from the fact that DA respects priorities when it is applied in Steps 1 and 2, and no student applies to an unacceptable school.

Example 5. *Let $S^1 = \{s_1\}$, $S^2 = \{s_2, s_3\}$, $I = \{i_1, i_2\}$, and $q_s = 1$ for all $s \in S$. Student preferences are given by: $s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} s_\emptyset$ and $s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} s_\emptyset$. The school priorities are given by: $i_1 \succ_{s_1} i_2 \succ_{s_1} \emptyset$, $i_2 \succ_{s_2} i_1 \succ_{s_2} \emptyset$, and $i_1 \succ_{s_3} i_2 \succ_{s_3} \emptyset$.*

Suppose first that both students report trutfully. In Step 1, i_1 and i_2 are assigned to s_1 and $s_\emptyset$, respectively. In Step 2, i_1 and i_2 are assigned to s_3 and s_2, respectively. In Step 3, i_1 and i_2 select s_3 and s_2, respectively. Hence, the seat at s_1 is wasted since i_2 prefers it to her assignment while the seat is unfilled.

Now, suppose that i_1 falsely reports s_1 as unacceptable. Then, in Step 1, i_1 and i_2 are assigned to $s_\emptyset$ and s_1, respectively. In Step 2, i_1 and i_2 are assigned to s_2 and $s_\emptyset$, respectively. In Step 3, i_1 and i_2 are assigned s_2 and s_1, respectively. Hence, i_1 gains by misreporting. □

In the following result, we restrict our attention to acyclic priority-capacity structures.[37] Moreover, for simplicity, we assume that all students are acceptable for all schools. We show that when the priority-capacity structure is acyclic, the decomposing DA mechanism is strategy-proof and its outcome is equivalent to the straightforward equilibrium outcome under rules of the game given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$.

Theorem 2. *(Revelation Principle) For any sequential problem with an acyclic priority-capacity structure where all students are acceptable for all schools, the decomposing DA mechanism is strategy-proof and its outcome under true preferences is equivalent to the straightforward SPNE outcome under $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$.*

7 Conclusions

This paper studies sequential school choice systems formulated as general multi-stage sequential games. One of the main takeaway messages of the paper is that there is a trade-off between the existence of a straightforward SPNE and non-wastefulness. More precisely, earlier studies have established that there are trade-offs that involve existence of equilibria and "efficiency notions" both in systems with parallel school assignments and in systems with sequential admissions (see, e.g., Doğan and Yenmez, 2019; Dur and Kesten, 2019; Ekmekci and Yenmez, 2019; Manjunath and Turhan, 2016). Most of these approaches exclusively focus on DA at the outset. What distinguishes our study from these papers is that the considered general sequential setting enables us to demonstrate that not only DA is the right choice but there is a unique set of rules for two-stage admission in school choice that guarantee the existence of a straightforward SPNE which, at the same time, reduces the waste of school seats as much as possible while respecting priorities.

Given the theoretical and empirical observations that manipulation is a problem in school choice, (see, e.g. Abdulkadiroğlu et al., 2005; Abdulkadiroğlu and Sönmez, 2013; Agarwal and Somain, 2014; Dur et al., 2018; Pathak and Sönmez, 2008, 2013), the results in this paper also

[37]Recall that, in Theorem 1 and Proposition 3, we show that an acyclic priority-capacity structure is needed to guarantee existence of straightforward SPNE when the rules of the game are given by $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$.

imply that there exists a rule that induces truthfulness (defined by the notion of straightforwardness) also for sequential settings. As already concluded in the above, using this rule comes at a cost in terms of wastefulness. However, this waste can be reduced by designing the school application sequence in a fashion such that students apply to either public or private schools first, depending on which set of schools that have the fewest number of (expected) applicants. The policy implication that comes out of this analysis is that if a social planner believes that truthful reporting is important, then the planner has to accept that school seats will be wasted but the waste can be reduced by a clever design.

Moreover, this unique design gives incentives to the schools assigned in Round 1 (private schools) to join the assignment in Round 2 (centralized public school admission).[38] Eventually, it will result in a unified admission system without coercing any school.

Because this paper is one of the first to study sequential school choice, there is room for future research. For example, the emphasis in this paper has been to introduce a notion of truthfulness in sequential school choice and to analyze the conditions needed for the existence of a straightforward SPNE in a very general framework and the implications on, e.g., wastefulness. Of course, there is a number of other axioms, notions and potential trade-offs that may be policy-relevant, and it is therefore difficult to draw too general conclusions based on this study alone. Hence, the findings and the analysis in this paper should be seen as a first step and not as a complete solution as more research is needed to fully understand complex sequential school choice systems.

Appendices

A The Extensive Form Game

This appendix gives a formal description of the extensive form game that previously was informally introduced in Section 3. The appendix first defines the extensive form game in general terms and then describes how this game fits in the considered sequential school admission framework.

An **extensive form matching mechanism** is a list $E = (I, H, M, \pi)$ where I is the set of **players**, H is the set of **histories** (nodes), M is the **strategy space**, and π is the **outcome function**.[39] The strategy space is composed of the **actions** that can be played at every history: $M = \Pi_{i \in I} M_i$ and $M_i = \Pi_{h \in H} M_i^h$ for every $i \in I$ where M_i^h is the actions that can be played by player $i \in I$ at history $h \in H$.[40] For a given **strategy profile** $m \in M$, we denote the strategy of player i with m_i. Hence, $m = (m_i)_{i \in I}$. Let h_1 be the **initial node** and H^T be the

[38]We prove this result in Proposition 7 Appendix C.

[39]We use a similar notation as Romero-Medina and Triossi (2014).

[40]Note that the analysis allows for the possibility that $M_i^h = s_\emptyset$ for some $i \in I$ and for some $h \in H$. In particular, $M_i^h = s_\emptyset$ means that player $i \in I$ is not active at node h.

set of **penultimate nodes**. The outcome function $\pi : H^T \rightarrow \mathcal{M}$ gives a matching for each penultimate node. Note that, given the initial node h_1, every strategy profile $m \in M$ defines a unique penultimate node. Let h_m be the penultimate node defined by strategy m. With a slight abuse of notation, for each strategy profile m, we use $\pi(m)$ instead of $\pi(h_m)$. A **preference profile** P and extensive form matching mechanism E constitute a **game** $\Gamma = (I, H, M, \pi, P)$.

Each node $h \in H \setminus H^T$ identifies a **subgame** $\Gamma(h) = (I, H(h), M(h), \pi_h, P)$ where h is an initial node, $H(h) = \{h' \in H | h' \text{ follows } h\}$, $M(h) = \Pi_{i \in I} \Pi_{h' \in H(h)} M_i^{h'}$, and $\pi_h(m) = \pi(\bar{h})$ where strategy $m \in M(h)$ specifies penultimate node $\bar{h} \in H^T$ starting from node h. Let $m(h) \in M(h)$ be the strategy of subgame $\Gamma(h)$ related with $m \in M$. A **Subgame Perfect Nash Equilibrium** (SPNE) is a strategy that induces a **Nash Equilibrium** in every (proper) subgame. That is, the strategy profile $m^* \in M$ is a subgame perfect Nash equilibrium if for all $h \in H$ and each player $i \in I$, it holds that $\pi_h(m^*(h)) R_i \pi_h(m'_i, m^*_{-i}(h))$ for every $m'_i \in \Pi_{h' \in H(h)} M_i^{h'}$.

We next describe how the above defined game can be translated into the sequential school admission framework considered in this paper. For this purpose, denote the histories in Round k of the sequential admission process by H^k. Because the admission process starts in Round 1, the initial node h_1 is unique and belongs to the set $H^1 = \{h_1\}$. Moreover, because the sequential admission process consists of three rounds, it follows that the set of penultimate nodes is given by $H^T = H^4$. Let the set of "active students" (discussed below) in node h be given by I^h. With slight abuse of notation, if $i \notin I^h$ for some $h \in H$, we suppose that $M_i^h = s_\emptyset$ (see also footnote 40). The set of active schools in Rounds 1 and 2 are given by S^1 and S^2, respectively. The sets of all possible (strict) rankings over $S^1 \cup \{s_\emptyset\}$ and $S^2 \cup \{s_\emptyset\}$ are given by $\mathcal{P}^1$ and $\mathcal{P}^2$, respectively. Consequently, the actions that can be played by any student $i \in I^{h_1}$ at node h_1 are given by $M_i^{h_1} = \mathcal{P}^1$. Similarly, the actions that can be played by any student $i \in I^h$ at node $h \in H^2$ are given by $M_i^h = \mathcal{P}^2$. Suppose now that $h \in H^k$, and let $h' = ((a_i)_{i \in I^h}, h) \in H^{k+1}$ be the node obtained from node h when each student $i \in I^h$ plays an action $a_i \in M_i^h$.

In the above, nothing has explicitly been stated about (i) the set of active students I^h at node $h \in H^2$ and (ii) the actions they can play at node $h \in H^3$. Both (i) and (ii) are determined by the **rules of the game**. As already explained in Section 3, these rules are given by the mechanisms ϕ^1 and ϕ^2 used in Rounds 1 and 2, respectively, and the correspondences ψ and γ. Here, the correspondence ψ determines the set of active students I^h at node $h = (a, h_1) \in H^2$. Note that the set of active students may depend on the Round 1 assignments that are determined by the mechanism $\phi^1(a)$. From Section 3, we know that the correspondence ψ prescribes that one of the following two cases prevail:

- $I^h = \{i \in I : \phi_i^1(a) = s_\emptyset\}$,
- $I^h(a) = I$.

The correspondence γ provides the actions that can be played by students in I^h at node $h \in H^3$, i.e., $\Pi_{i \in I^h} M_i^h$, where $M_i^h \subseteq \{\phi_i^1(\bar{a}), \phi_i^2(\hat{a})\}$ and $h = (\hat{a}, (\bar{a}, h_1))$. Moreover, the actions M_i^h

that a student i can take at node $h = (\hat{a}, (\bar{a}, h_1)) \in H^3$ depends on whether student i belongs to $I^{(\bar{a}, h_1)}$ or not. A more detailed description of the conditions (1) and (2a)–(2c) from Section 3 is as follows:

(1) if $i \notin I^{(\bar{a}, h_1)}$, then $M_i^h = \{\phi_i^1(\bar{a})\}$,

(2) if $i \in I^{(\bar{a}, h_1)}$, then one of the following three cases hold:

(a) $M_i^h = \{\phi_i^1(\bar{a}), \phi_i^2(\hat{a})\}$,

(b) $M_i^h = \begin{cases} \{\phi_i^1(\bar{a})\} & \text{if } \hat{a}_i = s_\emptyset \\ \{\phi_i^2(\hat{a})\} & \text{otherwise,} \end{cases}$

(c) $M_i^h = \begin{cases} \{\phi_i^1(\bar{a})\} & \text{if } \phi_i^2(\hat{a}) = s_\emptyset \\ \{\phi_i^2(\hat{a})\} & \text{otherwise.} \end{cases}$

Note that in case (2b), $\hat{a}_i = s_\emptyset$ is interpreted as not participating in Round 2 although student i is allowed to participate.

In the considered framework, a given problem $(S, I, q, P, \succ)$ and list of rules $(\phi^1, \phi^2, \psi, \gamma)$ induce the above described extensive form game and, consequently, the game $\Gamma = (I, H, M, \pi, P)$.

We end this appendix by stating a more formal definition of the notion of straightforwardness (see also Definition 3 in Section 3).

Definition 3. Given rules of the game $(\phi^1, \phi^2, \psi, \gamma)$, a student $i \in I$ plays a **straightforward** strategy m_i if:

- *Round* 1. For nodes $h \in H^1 = \{h_1\}$, the action m_i^h represents the true preferences of student i over the schools in $S^1 \cup \{s_\emptyset\}$.
- *Round* 2. For nodes $h = (\bar{a}, h_1) \in H^2$, the action m_i^h represents the true preferences of student i over the schools in $S^2 \cup \{s_\emptyset\}$ that are strictly preferred to $\phi_i^1(\bar{a})$ (note that $\phi_i^1(\bar{a})$ might be $s_\emptyset$).
- *Round* 3. For nodes $h \in H^3$, the action m_i^h for student i is always given by $m_i^h = \text{argmax}_{P_i} M_i^h$.

B Proofs

This appendix contains the proofs of all results presented in Sections 4 and 5.

Proof of Proposition 1. To prove the result, we provide examples in which either S^1 or S^2 is empty.[41]

[41] Note that one can easily modify these examples by adding schools to these sets.

Case $S^1 = \emptyset$. Suppose ϕ^2 is individually irrational <wasteful> [unfair]. Then, there exists a problem such that ϕ^2 selects a matching which is individually irrational <wasteful> [unfair]. Suppose that $\phi^2(I, S^2, q^2, \succ^2, P^2)$ is individually irrational <wasteful> [unfair]. Then, $(I, S, q, \succ, P) = (I, S^2, q^2, \succ^2, P^2)$ and for any (ψ, γ, ϕ^1) if there exists a straightforward SPNE the related equilibrium outcome, $\phi^2(I, S^2, q^2, \succ^2, P^2)$ is individually irrational <wasteful> [unfair].

Case $S^2 = \emptyset$. Suppose ϕ^1 is individually irrational <wasteful> [unfair]. Then, there exists a problem such that ϕ^1 selects a matching which is individually irrational <wasteful> [unfair]. Suppose that $\phi^1(I, S^1, q^1, \succ^1, P^1)$ is individually irrational <wasteful> [unfair] and $S^2 = \emptyset$. Then, $(I, S, q, \succ, P) = (I, S^1, q^1, \succ^1, P^1)$ and for any (ψ, γ, ϕ^2) if there exists a straightforward SPNE the related equilibrium outcome, $\phi^1(I, S^1, q^1, \succ^1, P^1)$ is individually irrational <wasteful> [unfair]. □

Proof of Proposition 2. The four parts of the proposition are proved one after another:

Part 1. Let $I = \{i\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, and $q = (1, 1)$. Suppose now that both schools regard student i as acceptable and that $s' P_i s P_i s_\emptyset$. Under the induced game, there exists a unique SPNE such that student i ranks $s_\emptyset$ above school s at the initial node h_1, student $i \in I^h$ ranks school s' over $s_\emptyset$ in any node $h \in H^2$ she participates, and student i selects the best option according to her true preferences in any node $h \in H^3$. This unique SPNE is not straightforward.

Part 2. Let $I = \{i, i'\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, $q = (1, 1)$, $i \succ_s i' \succ_s \emptyset$, $i \succ_{s'} i' \succ_{s'} \emptyset$, $s' P_i s_\emptyset P_i s$, and $s' P_{i'} s P_{i'} s_\emptyset$. Under the induced game, there exists a unique straightforward strategy profile: (i) student i ranks $s_\emptyset$ above s and student i' ranks school s above $s_\emptyset$ at the initial node h_1, (ii) both students rank school s' over $s_\emptyset$ in any node $h \in H^2$, and (iii) students i and i' select the best option according to their true preferences in any node $h \in H^3$. When both students play straightforward actions, student i' is matched to $s_\emptyset$ and she can profitably deviate by ranking $s_\emptyset$ over s' in any node $h \in H^2$.

Part 3. It will be demonstrated that the unique straightforward SPNE induces a wasteful outcome for the schools in S^1. Let $I = \{i, i'\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, $q = (1, 1)$, $i \succ_s i' \succ_s \emptyset$, $i \succ_{s'} i' \succ_{s'} \emptyset$, $s' P_i s P_i s_\emptyset$, and $s P_{i'} s_\emptyset P_{i'} s'$. Consider now the following strategy profile: (i) student i ranks school s above $s_\emptyset$ at the initial node h_1, school s' above $s_\emptyset$ at any node $h \in H^2$, and accepts her most preferred school at any node $h \in H^3$, and (ii) student i' ranks school s above $s_\emptyset$ at the initial node h_1, school $s_\emptyset$ above school s' at any node $h \in H^2$, and accepts only school s whenever possible at any node $h \in H^3$. This strategy profile is the unique straightforward SPNE, and in the related outcome, student i is matched to school s' and student i' is matched to $s_\emptyset$. Hence, the seat at school s is wasted for student i'.

Part 4. It will be demonstrated that a straightforward SPNE induces a wasteful outcome for the schools in S^2. Let $I = \{i, i'\}$, $S^1 = \{s\}$, $S^2 = \{s'\}$, $q = (1, 1)$, $i \succ_s i' \succ_s \emptyset$, $i \succ_{s'} i' \succ_{s'} \emptyset$, $s P_i s' P_i s_\emptyset$ and $s' P_{i'} s_\emptyset P_{i'} s$. Consider now the following strategy profile: (i) student i ranks school

s above $s_\emptyset$ at the initial node h_1, school s' above $s_\emptyset$ at any node $h \in H^2$,[42] and accepts her most preferred school at any $h \in H^3$, (ii) student i' ranks $s_\emptyset$ above school s at the initial node h_1, school s' above $s_\emptyset$ at any node $h \in H^2$, and accepts only school s' whenever possible at any $h \in H^3$. This strategy profile is a straightforward SPNE, and in the related outcome, student i is matched to school s and student i' is matched to school $s_\emptyset$. Hence, the seat at school s' is wasted for student i'. □

Proof of Theorem 1. Note first that if $i_1 \succ_{s_1} i_2 \succ_{s_2} i_3 \succ_{s_3} ... \succ_{s_n} i_1$, then there exist students $i, i' \in \{i_1, i_2, ..., i_n\}$ and schools $s, s' \in \{s_1, s_2, ..., s_n\}$ such that $i \succ_s i' \succ_{s'} i$. To see this, note that when considering the relative ranking between i_1 and i_3 at school s_2, there are two possibilities: (1) $i_1 \succ_{s_2} i_3$ or (2) $i_3 \succ_{s_2} i_1$. Then, we have either (1) $i_1 \succ_{s_2} i_3 \succ_{s_3} .... \succ_{s_n} i_1$ or (2) $i_1 \succ_{s_1} i_2 \succ_{s_2} i_1$. If the second case is true, then the desired relation is shown. If the first case is true, then we continue by considering the relative ranking between students i_1 and i_4 at school s_3.

We demonstrate that when $(\succ, q)$ is acyclic, there exists a straightforward SPNE for any problem. On the contrary, suppose that, at some problem, there exists a student i who benefits deviating from a straightforward strategy when all other students play a straightforward strategy. Let s be the school (possibly $s_\emptyset$) that student i gets when all students, including herself, play straightforward strategy, and let s' be the best school student i gets when all students except herself play a straightforward strategy (student i deviates).

Since ϕ^{1*} is strategy-proof, student i cannot get a better school in Round 1 by deviating. Therefore, $s' \in S_2$. Moreover, since ϕ^{2*} is strategy-proof, student i cannot get school s' by playing a straightforward action in Round 1 and by deviating in Round 2. Hence, student i can get school s' by playing a straightforward action in all subgames in Round 2 and by deviating from a straightforward action in Round 1. Finally, if such a strategy makes student i better off, then she can also be better off by ranking $s_\emptyset$ at the top in Round 1 and by acting straightforward in all subgames in Round 2. This follows from the fact that, under this strategy, other students get better choices in Round 1 and rank fewer schools in Round 2.

We consider two possible cases based on the assignment of student i in Round 1. In both cases, we use the sequential version of deferred acceptance mechanism (McVitie and Wilson, 1971) and allow student i to apply last (all other students are tentatively accepted by some school, including $s_\emptyset$). For simplicity, this mechanism is referred as the DA mechanism, henceforth. By Remark 2, the outcome of the DA mechanism in Round 1 is Pareto efficient with respect to the preferences over $S_1 \cup s_\emptyset$. That is, when student i applies to the schools that she finds better than her assignment in Round 1 under a straightforward action, she will be rejected. Otherwise, she causes a rejection chain, i.e., she causes some other student's rejection when it is her turn and that chain will end with her own rejection, leading to a Pareto inefficient assignment.

[42] Note that this strategy does not violate straightforwardness. Straightforwardness requires ranking schools better than first round assignment truthfully, but it does not bring any restrictions on the rankings of worse schools.

Case 1: Student i is unassigned in Round 1 when she plays a straightforward action. By the above observation, when student i applies to the schools that she finds acceptable, she will be rejected. Therefore, student i cannot cause any other student to be improved by deviating in Round 1. Therefore, straightforward strategy is an equilibrium.

Case 2: Student i is assigned in Round 1 when she plays straightforward action. Let school s_1 be student i's assignment in Round 1 under a straightforward strategy. Similar to Case 1, by the above observation, when student i applies to the schools that she finds better than s_1, she will be rejected. In particular, student i will be accepted by school s_1 and she will not be rejected in further steps. If student i does not cause any other student to be rejected from school s_1, student i cannot cause any other student to be improved by deviating in Round 1. Therefore, a straightforward strategy is an equilibrium. If student i causes some student to be rejected from school s_1, then a rejection chain starts. This chain ends up with some student that is unassigned or assigned to a school with available capacity. It is important to note that this chain is composed of distinct schools and students by Pareto efficiency. Denote this chain by:

$$i = i_0 \rightarrow s_1 \rightarrow i_1 \rightarrow \ldots \rightarrow s_n \rightarrow i_n.$$

Here, student i_k (for $k = 1, \ldots, n-1$) is rejected from school s_k due to student i_{k-1}'s application, and student i_n does not cause any rejection. Now, $i_k \succ_{s_{k+1}} i_{k+1}$ by definition of the DA mechanism. These students are the only students that are worse off by student i's application to school s_1. In particular, these students will apply to more schools in Round 2 compared to the case when student i states all schools in Round 1 as unacceptable.

Consider the case where student i plays straightforward in Round 2 and ranks $s_\emptyset$ at the top in Round 1. Again, we consider the sequential DA mechanism in Round 2. Furthermore, we allow all other students to apply to schools by using their straightforward list when student i manipulates in Round 1 and applies last. When it is student i's turn, she will only be accepted by school s'. Then we allow the students in $\{i_1, ..., i_n\}$ (i.e., the students included in the chain in Round 1) to apply to their remaining choices if they have not already been tentatively assigned by the DA mechanism. Since student i is not assigned to school s' when she plays straightforward strategy in Round 1, there exists a rejection chain starting with a student in $\{i_1, ..., i_n\}$. This chain leads to student i's rejection from school s'. Consider now the shortest such a chain. That is, let the chain start with a student in $\{i_1, ..., i_n\}$, include only one student in that chain, say i_k, and let the chain end with $s' \rightarrow i$. Then, we can connect that chain with $i \rightarrow s_1 \rightarrow i_1 \rightarrow ... \rightarrow s_k \rightarrow i_k$ and obtain a rejection cycle with distinct students and schools. Moreover, since these students are rejected by the schools, they don't have the highest q priority by the schools that rejects them. This contradicts weak acyclicity.

Note that, since ϕ_2^* is non-wasteful, if there exists an SPNE such that outcome is wasteful for the schools in S^2, then at least one school does not fill its capacity in Round 2 and at least one student does not report her true preferences. □

Proof of Proposition 3. We prove the result by constructing a problem. Let $\succ$ be a priority structure and q be a vector of capacities. We have distinct schools $s, s' \in S$ and students $i, j \in I$ such that:

- *Cycle condition:* $i \succ_s j \succ_s \emptyset$ and $j \succ_{s'} i \succ_{s'} \emptyset$.
- *Scarcity condition:* There exist (possibly empty) sets of agents $I_s, I_{s'} \subset I \setminus \{i, j\}$ such that $I_s \subset \{k \in I | k \succ_s j\} = U_s(j)$, $I_{s'} \subset \{k \in I | k \succ_{s'} i\} = U_{s'}(i)$, $|I_s| = q_s - 1$ and $|I_{s'}| = q_{s'} - 1$.

Let $s \in S_1$ and $s' \in S_2$. Let $|I_s|$ number of students in $U_s(j)$, including student i, consider school s as acceptable. Only $|I_{s'}|$ number of students in $U_{s'}(i)$, including student j, consider school s' as acceptable. If student k belongs to the set $U_s(j) \cap U_{s'}(i)$ and finds both schools acceptable, then she ranks school s' over school s. Student i ranks school s' over school s and finds all other schools unacceptable. Student j ranks school s over school s' and finds all other schools unacceptable. All the other students find schools s and s' unacceptable. Then, student i would like to deviate from playing a straightforward strategy when all other students play it. In particular, if all students play their straightforward strategy, then student i will be assigned to school s. On the other hand, if student i ranks all schools unacceptable in Round 1 while all other students play their straightforward strategy, then student i will be assigned to school s' in Round 2 by ranking it as her top choice. □

Proof of Proposition 4. Consider an arbitrary grand problem $(S, I, q, P, \succ)$, let m be an SPNE strategy profile, and μ be the induced equilibrium outcome.

It is first demonstrated that any SPNE is individually rational. To obtain a contradiction, suppose that μ is individually irrational. Then there exists a student i who prefers $s_\emptyset$ to her match μ_i. But then student i can be better off by ranking $s_\emptyset$ as her top choice at each node $h \in H^1 \cup H^2$ since she will be assigned to $s_\emptyset$ because an individually rational mechanism is used in both rounds. This contradicts that the strategy profile m is an SPNE.

It is next demonstrated that any SPNE is fair. We first focus on public schools. Suppose that there exist a student i and a public school s such that $sP_i\mu_i$ and $i \succ_s j$ for some $j \in \mu_s$. Then, at node h_1, student i can rank school s as her top choice and will, consequently, be assigned to school s. The latter conclusion follows from the fact that the constrained serial dictatorship mechanism is adopted. Consequently, whenever it is student i's turn, school s has an available seat and all remaining students have lower test scores than student i. This contradicts that m is an SPNE.[43]

Next consider private schools, and recall that under the old Turkish system, all students can participate in Round 2. Suppose now that there exist a student i and a private school s such that

[43]One can show that this result holds whenever $(\succ, q)$ is acyclic by considering the sequential deferred acceptance mechanism in which student i applies last.

$sP_i\mu_i$ and $i \succ_s j$ for some $j \in \mu_s$. As explained in the above, student i can then be assigned to school s by submitting the same action at node h_1 and by ranking school s as her top choice in any node $h \in H^2$. This contradicts that m is an SPNE.[44] □

Proof of Proposition 5. Consider an arbitrary grand problem $(S, I, q, P, \succ)$, let m be an SPNE strategy profile, and μ the induced equilibrium outcome.

It is first demonstrated that any SPNE is individually rational. To obtain a contradiction, suppose that μ is individually irrational. Then there exists a student i who prefers $s_\emptyset$ to her match μ_i. But then student i can be better off by ranking $s_\emptyset$ as her top choice at each node $h \in H^1 \cup H^2$ since individually rational mechanisms will assign her to $s_\emptyset$. This contradicts that the strategy profile m is an SPNE.

It is next demonstrated that any SPNE is non-wasteful and fair. We first focus on private schools. Suppose that there exist a student i and a private school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. In both cases, at node h_1, student i can rank school s as her top choice and will, consequently, be assigned to school s. The latter conclusion follows from the fact that the constrained serial dictatorship mechanism is adopted. Consequently, whenever it is student i's turn, school s has an available seat and all remaining students have lower test scores than student i. This contradicts that m is an SPNE.[45]

Next consider public schools. Suppose first that there exist a student i who participates in Round 2 under strategy profile m and a public school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. As explained in the above, student i can be assigned to school s by submitting the same action at node h_1 and by ranking school s as top choice in any node $h \in H^2$. Suppose instead that there does not exist a student i who participates in Round 2 under strategy profile m and a public school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. Instead, there exist a student i who does not participate in Round 2 under strategy profile m and a public school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. Consider now the strategy m'_i in which student i ranks $s_\emptyset$ as her top choice at node h_1, and school s as her top choice in any node $h \in H^2$ she participates. Let $m' = (m'_i, m_{-i})$. Under strategy profile m' student i is unassigned in Round 1 and she can, therefore, participate in Round 2. Moreover, the set of students participating in Round 2 under strategy profile m' is a subset of students participating in Round 2 under strategy-profile m. Then, since the constrained serial dictatorship mechanism is adopted in Round 2, in any node $h \in H^2$, school s will have an available seat when it is student i's turn. That contradicts that m is an SPNE.[46] □

Proof of Proposition 6. Consider an arbitrary grand problem $(S, I, q, P, \succ)$, let m be an SPNE

[44] One can show that this result holds whenever $(\succ, q)$ is acyclic by considering the sequential deferred acceptance mechanism in which student i applies last.

[45] One can show that this result holds whenever $(\succ, q)$ is acyclic by considering the sequential deferred acceptance mechanism in which student i applies last.

[46] One can show that this result holds whenever $(\succ, q)$ is acyclic by considering the sequential deferred acceptance mechanism in which student i applies last.

strategy profile, and μ the induced equilibrium outcome.

It is first demonstrated that any SPNE is individually rational. To obtain a contradiction, suppose that μ is individually irrational. Then there exists a student i who prefers $s_\emptyset$ to her match μ_i. But then student i can be better off by ranking $s_\emptyset$ as her top choice at each node $h \in H^1 \cup H^2$ since individually rational mechanisms will assign her to $s_\emptyset$. This contradicts that the strategy profile m is an SPNE.

It is next demonstrated that any SPNE is non-wasteful and fair for private schools. Suppose that there exist a student i and a private school s such that $sP_i\mu_i$ and either $|\mu_s| < q_s$ or $j \succ_s i$ for some $j \in \mu_s$. In both cases, at node h_1, student i can rank school s as her top choice and will, consequently, be assigned to school s. The latter conclusion follows from the fact that in both cases, student i will be among the top q_s students applying to school s according to $\succ_s$. This contradicts that m is an SPNE.

Finally, we focus on the public schools. Suppose that there exist a student i and public school s such that $sP_i\mu_i$ and $|\mu_s| < q_s$. We first consider the case such that student i is unassigned in Round 1 and participates in Round 2 under strategy m. In this case, student i can get school s by only changing her ranking in Round 2 by ranking s at the top for any $h \in H^2$ she participates. Call this strategy m'_i and let $m' = (m'_i, m_{-i})$. Since the rankings submitted in Round 1 are unchanged, m will lead to the same subgame in Round 2. By using the sequential version of the deferred acceptance algorithm (McVitie and Wilson, 1971), one can easily see that under strategy profile m', i is assigned to s.

Now consider the case where student i is assigned in Round 1. Recall that in order to participate in Round 2, i needs to be unassigned in Round 1. Then, consider the following strategy m'_i for i: rank $s_\emptyset$ over all schools at h_1 and rank s over all schools at any node $h \in H^2$ she participates. Let $m' = (m'_i, m_{-i})$. Under strategy m' let h'^2 be the node reached in Round 2. Then, the set of students active in node h' will be a subset of the active students under m union i. If we restrict them to playing their weakly undominated actions, i.e., their true preferences over the public schools then, by using the sequential deferred acceptance algorithm, and the fact that the deferred acceptance algorithm is population monotonic we can show that when it is student i's turn she can get school s. □

Proof of Theorem 2 We first focus on the acyclic priority-capacity structures. Notice that if a priority-capacity structure $(\succ, q)$ is acyclic, then $i \succ_s j \succ_{s'} i$ implies either j has a top q_s priority under $\succ_s$ or i has a top $q_{s'}$ priority under $\succ_{s'}$. Suppose that the first case holds. Then, it is easy to see that swapping i and j under the priority order of s will not change the outcome of DA in both steps. In fact, we can achieve a priority profile in which all school rank students in the same order. Hence, in the rest of the proof, we consider a common priority order for all schools such that student i_k is ranked k^{th} in that order.

We prove that the separated DA mechanism is strategy-proof by induction. We start by considering i_1. It is easy to see that, when i_1 reports her true preference order, she will be assigned to her top choice among schools in $S^1 \cup s_\emptyset$ in Step 1. Let s_1 be that school. Similarly, in Step 2, she will be assigned to her top choice among schools in $S_2 \cup s_\emptyset$ that are ranked above s_1. As a result, she can pick her top choice in Step 3. Hence, i_1 cannot do better than reporting her true preference order. Notice that, students ranked below i_1 cannot affect her assignment.

Suppose each student i_k cannot do better than reporting her true preference order, she can be assigned to best available school once the Steps 1 and 2 assignments of higher ranked students are removed, and her assignment cannot be affected by the reports of students ranked below her. Now, we consider student i_{k+1}. When i_{k+1} reports her true preference order, she will be assigned to her top choice among available schools in S^1 and S^2 in Steps 1 and 2 when the corresponding assignments of higher ranked students are removed, respectively. As a result, she can pick her best remaining choice in Step 3. Hence, i_{k+1} cannot do better than reporting her true preference order. Notice that, students ranked below i_{k+1} cannot affect her assignment.

Finally, we show the outcome equivalence with the straightforward SPNE outcome under $(\phi^{1*}, \phi^{2*}, \psi^*, \gamma^*)$. First, in Round 1 of the game, notice that under any straightforward SPNE students rank their true preferences over the acceptable schools in S^1. As a result, in Round 1, any straightforward SPNE results in the same matching obtained under Step 1 of separated DA mechanism when students report their true preferences. Then, the preference order considered under the second step of the separeted DA and action played under the corresponding straightforward SPNE in Round 2 are the same. Hence, the students will select from the same set of options at the end of the game and the separated DA mechanism. □

C Incentives to Join Centralized Assignment

Propositions 1 and 2 illustrate that it is impossible to construct a sequential assignment procedure under which students play straightforward strategies and school seats are not wasted. On the other hand, under a simultaneous assignment procedure, in which all schools are available at the same time, it is possible to achieve desirable properties. For instance, when all schools are available at the same time, the DA mechanism is stable and strategy-proof, while the Top Trading Cycles (TTC) mechanism is Pareto efficient and strategy-proof. Hence, one immediate remedy to the school choice with public and private schools is to run a centralized assignment procedure where all schools are available for all students at the same time. However, even if public schools are controlled by a central authority, private schools can often not be forced to join the centralized assignment procedure (see the Introduction). Therefore, if a single round simultaneous assignment is desired, then private schools must be incentivized to join a centralized system. In the following proposition, we show that if all schools have the same relative priorities over acceptable students, the rules of the admission system is given by $(SD, SD, \psi^*, \gamma^*)$ and the private school admission is done first, then each private school would like to join the centralized admission in Round 2.[47]

Proposition 7. *Suppose that all schools in S have the same relative priorities over acceptable students. Let $(SD, SD, \psi^*, \gamma^*)$ be the rules of the admission system. Suppose all students play straightforward actions. For any problem $((S^{pu}, S^{pr}), I, q, P, \succ)$, let μ and ν be the outcomes of this system when private school $s \in S^{pr}$ participates in the first round and the second round while keeping everything else the same, respectively. Then, $\nu_s \succsim_s \mu_s$.* [48]

Proof. In the rest of the proof, we refer to the case in which all private schools participate in Round 1 as Case 1, and the case in which only private school s participates in Round 2 as Case 2. Let μ^1 and ν^1 be the matchings selected in the first round of Case 1 and Case 2, respectively. When all students play straightforward actions, the outcome in the first round is equivalent to the outcome of the SD mechanism under true preferences over the available schools in the first round. It is easy to see that when the number of available schools decreases, all students become (weakly) worse off under the serial dictatorship mechanism. Hence, under Case 2, all students in μ^1_s are assigned to a worse private school than s in ν^1. All the other students become weakly worse off compared to μ^1.

On the contrary, suppose $\mu_s \succ_s \nu_s$. Then, there exists at least one student $\bar{i}$ such that $\mu_{\bar{i}} = s$, $\nu_{\bar{i}} \neq s$ and $\nu_{\bar{i}} P_{\bar{i}} \mu_{\bar{i}}$. To see this, we consider the following cases in which our claim does not hold: **Case (a)** $\nu_i = s$ for each $i \in \mu_s$, and **Case (b)** $\mu_s \neq \nu_s$ and $\mu_i R_i \nu_i$ for all $i \in \mu_s$. If Case (a) is true, then $\mu_s \subseteq \nu_s$. Since all students in ν_s are acceptable for s, $\nu_s \succsim_s \mu_s$. If Case (b) is true, then under ν school s fills all its available seats, i.e., $|\nu_s| = q_s$, and there exists at least one student $i \in \mu_s$ such that $\mu_i P_i \nu_i$. Without loss of generality, let i have the highest priority among such students. That is, if $j \in \mu_s$ and $j \succ_s i$, then $\nu_j = s$. Moreover, all students in ν_s have higher priority than i. Therefore, $\nu_s \succ_s \mu_s$.

Without loss of generality suppose that student i is the student with the highest priority such that $\mu_i = s$ but $\nu_i \neq s$ and $\nu_i P_i s$. Let $s' = \nu_i$. Since school s is the only private school participating in Round 2 under Case 2, school s' needs to be a public school. Since $\mu_i R_i \mu^1_i$ and $s' P_i \mu_i$, under Case 1 student i also applies to school s' in Round 2. Since student i is not assigned to school s' in Case 1, all assignees of s' have higher priority than student i and $|\mu_{s'}| = q_{s'}$. Then, there exists at least one student who is assigned to s' in μ but not in ν. Let $j \in (\mu_{s'} \setminus \nu_{s'})$. Then, student j has higher priority than student i and $s' P_j s$. Then student j has to be assigned to a better public school than s' under ν. If we apply the same reasoning for student j, then due to the finite number of students and schools, we will reach a contradiction. That is, there does not exist an $i \in \mu_s$ but $i \notin \nu_s$ and $\nu_i P_i s$. Hence, if $i \in \mu_s$ but $i \notin \nu_s$ then $s P_i \nu_s$. That is, all slots of school s are filled with students who have higher priority than student i. This means that a better set of students is assigned to school s under Case 2 and $\nu_s \succsim_s \mu_s$. □

[47] Recall that when all schools have the same relative priorities over acceptable students, SD and DA are outcome equivalent.

[48] When there is no restriction on the priorities over acceptable students, Ekmekci and Yenmez (2019) provide a negative result.

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